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On the Projective Differential Classification of N-Dimensional Spreads Generated by ∞^1 Flats.

BY ARTHUR RANUM.

INTRODUCTION.

This paper is a continuation of the investigation begun in my former paper in the *Annali di Matematica Pura ed Applicata*, Vol. XIX (1912), pp. 205–249, "On the Projective Differential Geometry of N-Dimensional Spreads Generated by ∞^1 Flats." The chief goal aimed at here is the classification of m-spreads generated by a single infinity of $(m-1)$-flats (linear spreads) in n-dimensional space from the projective differential standpoint.

The methods developed and apparatus set up in the earlier paper, and more particularly those in Part I, are here applied to the problem of classifying these m-spreads by means of their most fundamental projective differential properties. So far as four and five dimensions are concerned, this classification is carried out with considerable detail in §§ 39–73. For the higher spaces the broader outlines alone are given (§§ 33–38). The first half of the paper (§§ 1–32) contains a variety of somewhat disconnected developments of a general nature, but all have a more or less direct bearing on the main subject that follows and are necessary as preparing the way for it. For the sake of brevity the earlier paper will be referred to as "P. D. G."

Very recently a paper by E. Bompiani has appeared in the *Rendiconti di Palermo*, Vol. XXXVII (1914), pp. 305–331, entitled "Alcune proprietà proiettivo-differenziali dei sistemi di rette negli iperspazi," which has rather close contact with the part of my paper dealing with ruled surfaces in F_4. He has there introduced two very useful numbers called the *indices of developability*, the second of which corresponds exactly to my classes (a), (b) and (c). He has not, however, carried out the classification so completely as I have.*

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* In the *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 89–110, I have made a further contribution to the theory of ruled surfaces and planar three-spreads in four-space, based on §§ 42–59 of this paper, and have made an application of the results to the sphere-geometry of cyclic surfaces in three-space.

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I. ENCLOSED SPREADS.

1. Enclosed spreads were defined in "P. D. G.", § 17; about the only cases considered in that paper were those in which one spread encloses another belonging to the same tree. We now consider the general case.

Let $S = S^r_m$ enclose $\bar{S} = S^{\bar{r}}_{\bar{m}}$. Then the tangent spread $S^1 = S_{m+r}$ encloses $\bar{S}^1 = S_{\bar{m}+\bar{r}}$, and the focal spread $S_{1,0} = S_{m-r}$ encloses $\bar{S}_{1,0} = S_{\bar{m}-\bar{r}}$. Hence, the conditions

$$\left.\begin{array}{c} m \geqq r, \qquad \bar{m} \geqq \bar{r}, \qquad m \geqq \bar{m}, \\ m + r \geqq \bar{m} + \bar{r}, \qquad m - r \geqq \bar{m} - \bar{r}, \end{array}\right\} \tag{1}$$

and therefore

$$m - \bar{m} \geqq r - \bar{r} \geqq -(m - \bar{m}), \tag{2}$$

must be satisfied. It follows that *while the range of a spread may be greater or less than that of an enclosed spread, the difference between their ranges can not be greater than the difference between their dimensionalities.* It is easy to show by concrete examples that $r - \bar{r}$ can have any integral value from $m - \bar{m}$ to $-(m - \bar{m})$.

If $r - \bar{r} = m - \bar{m}$ and S^r_m encloses $S^{\bar{r}}_{\bar{m}}$, their focal spreads coincide.

If $r - \bar{r} = -(m - \bar{m})$ and S^r_m encloses $S^{\bar{r}}_{\bar{m}}$, their tangent spreads coincide.

2. If S is given and the conditions (1) are satisfied, there does not necessarily exist any spread $\bar{S}$ enclosed in it. For instance, if a spread S^2_4 expressed in normal form ("P. D. G.," § 44) is $[A_1, A_2; A'_1, A'_2]$, it encloses an infinite number of spreads S^3_3 and S^2_3, but no S^1_3 and no S^0_2.

But it is easy to see that

if the focal spread of a given S^r_m is of range s, and if $r - \bar{r} = m - \bar{m}$ and $r \geqq \bar{r} \geqq s$, there exists an $S^{\bar{r}}_{\bar{m}}$ enclosed in S^r_m. In particular, putting $\bar{r} = s$, we find that *an S^r_m whose focal spread is of range s has just one enclosed spread of dimensionality $m - r + s$ and range s*, and that is its special enclosed spread.*

if the tangent spread of a given S^r_m is of range t, and if $r - \bar{r} = \bar{m} - m$ and $r \geqq \bar{r} \geqq t$, there exists an $S^{\bar{r}}_{\bar{m}}$ enclosing S^r_m. In particular, putting $\bar{r} = t$, we find that *an S^r_m whose tangent spread is of range t has just one enclosing spread of dimensionality $m + r - t$ and range t*, and that is its special enclosing spread.*

More generally, an S^r_m whose first i focal spreads are of ranges $r_1, \ldots, r_i$, respectively, has only one enclosed spread of dimensionality $m - r - r_1 - \ldots - r_{i-1} + ir_i$ and range r_i, namely its i-th special enclosed spread.

More generally, an S^r_m whose first i tangent spreads are of ranges $r_1, \ldots, r_i$, respectively, has only one enclosing spread of dimensionality $m + r + r_1 + \ldots + r_{i-1} - ir_i$ and range r_i, namely its i-th special enclosing spread.

Common Enclosed Spreads.

3. We define the *common enclosed spread* of two given spreads as the spread of highest dimensionality enclosed in them. It is the locus of the flats of intersection of their corresponding generators. Its directrices are those which are common to the two given spreads.

We define the *common enclosing spread* of two given spreads as the spread of lowest dimensionality enclosing them. It is the locus of the connecting flats of their corresponding generators. Its directrices are those curves which are linearly dependent on the directrices of the two given spreads.

Notice that the common enclosed spread is not the locus of all the common points of the two given spreads, but only of those which are common to the pairs of corresponding generators. It depends not merely on the given spreads themselves, but also on the form of their equations; that is, on the correspondence that has been established between their generators by the choice of the parameter ω in both cases. If the equations of one of the given spreads are altered by a transformation to a new parameter,

$$\bar{\omega} = f(\omega),$$

a new correspondence is set up, and a new common enclosed (enclosing) spread is obtained. In the later applications (§§ 8–12, 42–73) to the case where the given spreads belong to the same tree, this element of arbitrariness is not present.

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* See "P. D. G.," § 31. Here "special enclosed (enclosing) spread" means what is there called "*first* special enclosed (enclosing) spread."

The sum of the dimensionalities of the common enclosed spread and the common enclosing spread is obviously equal to the sum of the dimensionalities of the given spreads.[*]

In view of "P. D. G.," § 29, we see that

if U is the common enclosed spread of S and T, and if $\overline{U}$, $\bar{S}$ and $\bar{T}$ are corresponding spreads of their respective trees, then $\overline{U}$ is enclosed in $\bar{S}$ and in $\bar{T}$, and therefore also in their common enclosed spread.

if V is the common enclosing spread of S and T, and if $\overline{V}$, $\bar{S}$ and $\bar{T}$ are corresponding spreads of their respective trees, then $\overline{V}$ encloses $\bar{S}$ and $\bar{T}$, and therefore also their common enclosing spread.

4. But we can say more than this. Let

$$U = [A_1, \ldots, A_a];$$

then we can put

$$S = [A_1, \ldots, A_a;\ B_1, \ldots, B_b],$$
$$T = [A_1, \ldots, A_a;\ C_1, \ldots, C_c],$$

and

$$V = [A_1, \ldots, A_a;\ B_1, \ldots, B_b;\ C_1, \ldots, C_c],$$

choosing the fundamental directrices in such a way that they are independent in all four cases. The fundamental directrices of the tangent spread

$$U^1 = [A_1, \ldots, A_a;\ A'_1, \ldots, A'_a]$$

are not, in general, independent; let a' be the number of independent linear relations connecting them; then, $0 \leqq a' \leqq a$. S^1, T^1 and V^1 will be affected by these same relations and also, in general, by others. Let $a' + b'$ be the number of independent relations connecting the fundamental directrices of S^1, and $a' + c'$ the number connecting those of T^1; then,

$$0 \leqq b' \leqq 2b, \qquad a' + b' \leqq a + b,$$
$$0 \leqq c' \leqq 2c, \qquad a' + c' \leqq a + c.$$

The fundamental directrices of V^1 will be connected by the $a' + b' + c'$ independent relations already mentioned, and also, in general, by d' others that are independent of one another and of the former relations. Evidently,

$$a' + b' + c' + d' \leqq a + b + c.$$

These latter d' relations are of the form

$$\phi(\ldots, A'_i, A_i, \ldots) + \chi(\ldots, B'_i, B_i, \ldots) + \psi(\ldots, C'_i, C_i, \ldots) = 0, \tag{3}$$

where neither the χ-function nor the ψ-function is identically zero.

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[*] This holds even when one or more of the spreads is an $S_m{}^0 = F_{m-1}$, provided the word "dimensionality" be here taken to stand for the number m. See "P. D. G.," § 9.

The ranges of the four spreads U, S, T, V are equal to

$$a - a', \quad a - a' + b - b', \quad a - a' + c - c', \quad a - a' + b - b' + c - c' - d',$$

respectively; and the sum of the ranges of S and T is exactly d' units greater than the sum of the ranges of U and V. Hence, *the sum of the ranges of two spreads can not be less than the sum of the ranges of their common enclosed spread and their common enclosing spread.*

5. V^1 is obviously the common enclosing spread of S^1 and T^1; and ascending higher, we see that V^2 is the common enclosing spread of S^2 and T^2, etc. This theorem and its correlative may be stated as follows:

If U is the common enclosed spread of S and T, and if U_i, S_i and T_i are their respective i-th focal spreads, then U_i *coincides with the common enclosed spread of* S_i *and* T_i.	If V is the common enclosing spread of S and T, and if V^i, S^i and T^i are their respective i-th tangent spreads, then V^i *coincides with the common enclosing spread of* S^i *and* T^i.

6. On the other hand, the common enclosed spread of S^1 and T^1 encloses U^1, but does not, in general, coincide with it. By reason of the relations (3) it also encloses precisely d' directrices that are independent of each other and of those of U^1; and its dimensionality is d' units greater than that of U^1.

By the principle of "P. D. G.," § 16, every identical relation affecting S^1 gives rise to a directrix of the focal spread S_1 and *vice versa.* Hence, the dimensionalities of U_1, S_1, T_1, V_1 are equal to a', $a' + b'$, $a' + c'$, $a' + b' + c' + d'$, respectively; and the dimensionality of the common enclosing spread of S_1 and T_1 is $a' + b' + c'$, which is d' units less than that of V_1.

Comparing the last two paragraphs, we see

that the tangent spread U^1 of the common enclosed spread of S and T is enclosed in the common enclosed spread $\overline{U}$ of their tangent spreads,	that the focal spread V_1 of the common enclosing spread of S and T encloses the common enclosing spread $\overline{V}$ of their focal spreads,

and that *the difference between the dimensionalities of* $\overline{U}$ *and* U^1 *is equal to the difference between the dimensionalities of* $\overline{V}$ *and* V_1. Hence, if U^1 coincides with $\overline{U}$, V_1 coincides with $\overline{V}$, and conversely. A simple example in which U^1 does not coincide with $\overline{U}$ occurs in the discussion of ruled surfaces in F_4 (§§ 45–48), and is obtained by taking the spreads S_2^2 and S_2^1 for S and T, respectively.

The Extended Tree.

7. If two spreads belong to the same tree, their common enclosed (enclosing) spread may or may not belong to the tree. If not, it determines a *secondary tree* which may, however, have some spreads in common with the

primary tree. The totality of the spreads of a tree together with those of all its secondary trees we shall call an *extended tree.*

Secondary Trees Determined by Common Enclosing Spreads.

8. Following the notation of "P. D. G.," §§ 39–46, let us consider a spread S expressed in its normal form. By merely inspecting the array (24), of § 44, we can easily learn the structure of a large number of the spreads of the extended tree determined by S. Thus the two spreads $S_{\alpha\beta}$ and $S_{\lambda\mu}$,* for which

$$\alpha \geqq \beta \text{ and } \lambda \geqq \mu, \tag{4}$$

belong to the respective ascending series determined by two of the focal spreads of S, and are both enclosed in S. The fundamental directrices of each of these two spreads are those situated in a rectangle in the lower left-hand corner of the array (24). Their common enclosed spread is therefore another spread of the same kind, and belongs to the primary tree. Their common enclosing spread, however, will give us something new, provided neither spread encloses the other. To ensure this we first put

$$\lambda > \alpha, \quad 0 < r_{t-\lambda} < r_{t-\alpha}, \tag{5}$$

so that the two spreads may belong to distinct ascending series; $S_{\lambda\mu}$ can not then enclose $S_{\alpha\beta}$. Moreover, if

$$\alpha - \beta > \lambda - \mu, \tag{6}$$

$S_{\alpha\beta}$ can not enclose $S_{\lambda\mu}$.†

Hence, if the conditions (4), (5) and (6) are satisfied, the common enclosing spread T of $S_{\alpha\beta}$ and $S_{\lambda\mu}$ will belong to the extended tree determined by S without belonging to the primary tree. T is evidently of range $r_{t-\alpha}$. By varying $\alpha, \beta, \lambda, \mu$ in all possible ways, we obtain a fourfold system W of such spreads.

9. Consider the secondary tree determined by T. By the right-hand theorem of § 5, T^j is the common enclosing spread of $(S_{\alpha\beta})^j = S_{\alpha, \beta+j}$ and $(S_{\lambda\mu})^j = S_{\lambda, \mu+j}$. Hence, for $j = 1, \ldots, \lambda - \mu$, T^j belongs to the system W, and is of range $r_{t-\alpha}$. For higher values of j, a complete discussion of T^j would involve a consideration of the identical linear relations connecting its fundamental directrices, and we shall not go into it.

It is also clear that for $i = 1, \ldots, \beta$, T_i is the common enclosing spread of $(S_{\alpha\beta})_i = S_{\alpha, \beta-i}$ and $(S_{\lambda\mu})_i = S_{\lambda, \mu-i}$, and is of range $r_{t-\alpha}$ and belongs to W. Hence, the first $\lambda - \mu$ successive tangent spreads of T and its first β successive focal spreads, together with T itself, form a regular series of $\beta + \lambda - \mu + 1$ spreads of range $r_{t-\alpha}$.

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* See "P. D. G.," § 29.

† Cf. "P. D. G.," § 30.

Continuing the descending series, we find that for $i = \beta + 1, \ldots, \beta + \lambda - \alpha - 1$, T_i is the common enclosing spread of $(S_{\alpha\beta})_i = S_{\alpha-\beta+i,0}$ and $(S_{\lambda\mu})_i = S_{\lambda,\mu-i}$, and belongs to W, if its range $r_{t-\alpha+\beta-i} > r_{t-\lambda}$. Otherwise it coincides with $S_{\lambda,\mu-i}$ and belongs to the primary tree. Further on, for $i = \beta + \lambda - \alpha, \ldots, \mu$, T_i surely coincides with $S_{\lambda,\mu-i}$, and is of range $r_{t-\lambda}$. In the final stage of the descent, for which $i = \mu + 1, \ldots, t - \lambda + \mu$, $T_i = S_{\lambda-\mu+i}$ and is of range $r_{t-i-\lambda+\mu}$. That is, the descending series determined by T here coalesces with that determined by S, although the corresponding terms do not coincide. The last spread of the series is $T_{t-\lambda+\mu} = S_t$, which is, naturally, of range $r_0 = 0$.

We see, therefore, that the descending series determined by T, consisting of $t - \lambda + \mu + 1$ distinct spreads, comprises *four radically different parts.* The first part is a regular series of $\beta + 1$ terms; the second part has $\lambda - \alpha - 1$ terms and is irregular; the third part is again regular, with $\alpha - \beta - \lambda + \mu + 1$ terms; and the fourth part is again irregular, with $t - \lambda$ terms.

10. Going back to a spread T_i belonging to the second part of this series, where $\beta < i < \beta + \lambda - \alpha$, we see that its j-th tangent spread T_{ij}, for all values of j, is the common enclosing spread of $(S_{\alpha\beta})_{ij} = S_{\alpha-\beta+i,j}$ and $(S_{\lambda\mu})_{ij} = S_{\lambda,\mu-i+j}$. For values of j not greater than $\lambda - \mu + i$, it is of range $r_{t-\alpha+\beta-i}$ and belongs, in general, to W.

It is now clear that all those spreads of W for which λ and $\alpha - \beta - \lambda + \mu$ $(= \varkappa)$ are constant, and form, therefore, a twofold system, *belong to the same secondary tree,* and that this secondary tree is determined by the common enclosing spread of $S_{\varkappa,0}$ and $S_{\lambda\lambda}$, where

$$0 < \varkappa < \lambda \quad \text{and} \quad 0 < r_{t-\lambda} < r_{t-\varkappa}. \tag{7}$$

By varying λ and $\varkappa$ in all possible ways, subject to the condition (7), we obtain a twofold system of secondary trees determined by S.

Secondary Trees Determined by Common Enclosed Spreads.

11. By merely applying the principle of duality, we can discuss in a similar manner an entirely different part of the extended tree determined by a spread S. The results are briefly as follows: Let the spreads $S^{\alpha\beta}$ and $S^{\lambda\mu}$ satisfy the conditions (4), (6) and

$$\lambda > \alpha, \quad 0 < r_{t+\lambda} < r_{t+\alpha}.^* \tag{5'}$$

Their common enclosed spread T is of range $r_{t+\alpha}$ and does not belong to the primary tree determined by S. T_j is the common enclosed spread of $S^{\alpha,\beta+j}$

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* As in "P. D. G.," § 68, $r_{t+\lambda}$ denotes the range of S^{λ}.

and $S^{\lambda,\mu+j}$, and, for $j=1,\ldots,\lambda-\mu$, is of range $r_{t+\alpha}$ and belongs to the system W of spreads T. T^i $(i=1,\ldots,\beta)$ is the common enclosed spread of $S^{\alpha,\beta-i}$ and $S^{\lambda,\mu-i}$, and is of range $r_{t+\alpha}$ and belongs to W. Hence, the first $\lambda-\mu$ successive focal spreads and the first β successive tangent spreads of T, together with T itself, form a regular series of range $r_{t+\alpha}$.

Continuing the ascending series, we see that for $i=\beta+1,\ldots,\beta+\lambda-\alpha-1$, T^i is the common enclosed spread of $S^{\alpha-\beta+i,0}$ and $S^{\lambda,\mu-i}$ and is of range $r_{t+\alpha-\beta+i}$; for $i=\beta+\lambda-\alpha,\ldots,\mu$, T^i coincides with $S^{\lambda,\mu-i}$ and is of range $r_{t+\lambda}$; and for $i=\mu+1,\ldots,u-\lambda+\mu$, $T^i=S^{\lambda-\mu+i}$ and is of range $r_{t+i+\lambda-\mu}$. The last spread of the series, $T^{u-\lambda+\mu}=S^u$, is of range $r_{t+u}=0$. Hence, the ascending series determined by T consists of four parts, containing $\beta+1$, $\lambda-\alpha-1$, $\alpha-\beta-\lambda+\mu+1$ and $u-\lambda$ spreads, respectively; *the first and third of these parts are regular, and the second and fourth irregular.*

If $i=\beta+1,\ldots,\beta+\lambda-\alpha-1$, and $j=1,\ldots,\lambda-\mu+i$, T^{ij} is the common enclosed spread of $S^{\alpha-\beta+i,j}$ and $S^{\lambda,\mu-i+j}$, and is of range $r_{t+\alpha-\beta+i}$ and belongs to W. Those spreads of the fourfold system W for which λ and $\alpha-\beta+\lambda-\mu$ $(=\varkappa)$ are constant belong to the same secondary tree, which is determined by the common enclosed spread of $S^{\kappa,0}$ and $S^{\lambda\lambda}$, where

$$0<\varkappa<\lambda \quad\text{and}\quad 0<r_{t+\lambda}<r_{t+\kappa}. \tag{7'}$$

By varying λ and $\varkappa$, we obtain a twofold system of secondary trees determined by S.

12. Incidentally, it is clear that

if T is the common enclosing spread of $S_{\alpha\beta}$ and $S_{\lambda\mu}$, satisfying (4), T_{ij} is the common enclosing spread of $(S_{\alpha\beta})_{ij}$ and $(S_{\lambda\mu})_{ij}$, for all values of i and j.

if T is the common enclosed spread of $S^{\alpha\beta}$ and $S^{\lambda\mu}$, satisfying (4), T^{ij} is the common enclosed spread of $(S^{\alpha\beta})^{ij}$ and $(S^{\lambda\mu})^{ij}$, for all values of i and j.

II. THE ORDER OF AN ALGEBRAIC SPREAD.

13. If an m-spread S_m generated by ∞^1 $(m-1)$-flats in space of $n-1$ dimensions is algebraic, and if its range is >0, its *order* is the number of points it has in common, in general, with a fixed $(n-m-1)$-flat F_{n-m-1}. This $(n-m-1)$-flat must and can always be so chosen as to meet no generator in more than one point. Hence, the order of S_m is simply the number of its generators which meet F_{n-m-1}.

This definition is somewhat broader than the usual one, because it includes the case $m=n-1$, in which S_m, being generated by $(n-2)$-flats, is not a spread in the ordinary sense of the word. In this case its order is the number of generators that pass through a fixed point F_0.

In the simple case in which S_1 is a plane curve and S_2 its tangent spread (the locus of its tangents), their respective orders, as so defined, agree with what are ordinarily called the order and class of the curve. If S_1 is a curve immersed in a three-flat, S_2 its tangent surface and S_3 its second tangent spread (the locus of its osculating planes), their respective orders are what Salmon calls the order, rank and class of the entire system.* It is evident that, in the theory of the algebraic spread, not only its order but the orders of its tangent and focal spreads, and indeed of all the spreads of its tree, must be taken into account.

The Order of a Rational Spread.

14. Let us now confine ourselves to the case in which $S_m = [A_1, \ldots, A_m]$ is a rational spread. Then the equations of every directrix A_i can be written in the form

$$X_1 = A_{i1}(\omega), \ldots, X_n = A_{in}(\omega),$$

where $A_{i1}(\omega), \ldots, A_{in}(\omega)$ are rational integral functions of the parameter ω having no common factor in ω. Let the fundamental directrices $A_1, \ldots, A_m$ be independent; then the m-rowed determinants of the matrix

$$\| A_{ij}(\omega) \| \qquad (i = 1, \ldots, m; \ j = 1, \ldots, n) \tag{8}$$

can not all vanish identically. Let $F_{n-m-1} = [C_1, \ldots, C_{n-m}]$ be the fixed $(n-m-1)$-flat referred to in § 13, and let the constant coordinates of C_i be $C_{i1}, \ldots, C_{in}$. A necessary and sufficient condition that F_{n-m-1} meet a generator of S_m is

$$\begin{vmatrix} A_{11}(\omega) & \ldots & A_{1n}(\omega) \\ \cdots & \cdots & \cdots \\ A_{m1}(\omega) & \ldots & A_{mn}(\omega) \\ C_{11} & \ldots & C_{1n} \\ \cdots & \cdots & \cdots \\ C_{n-m,1} & \ldots & C_{n-m,n} \end{vmatrix} = 0.$$

Using Laplace's expansion in terms of the first m rows, we obtain an equation whose degree in ω gives the number of generators of S_m that meet F_{n-m-1}. We must, however, exclude those indeterminate generators which result from the vanishing of a possible common factor of all the m-rowed determinants of (8); for those particular values of ω the corresponding points of the directrices A_i are not independent. Hence, *the order of S_m is equal to the highest degree in ω of the expanded m-rowed determinants of the matrix* (8), *after the removal of their common factors.*

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* "Analytic Geometry of Three Dimensions," Vol. I (1912), Art. 325, p. 336.

III. PROJECTION AND SECTION.

Restricted and Conical Spreads.

15. If an m-spread in $(n-1)$-dimensional space is enclosed in a q-flat but not in a $(q-1)$-flat, it is said to be *immersed* in the q-flat. If $q < n-1$, it is a *restricted* spread.

If a spread S encloses a p-flat F_p but no $(p+1)$-flat, we shall call it a *conical spread* (or *cone*) whose *vertex* is F_p. If $p = 0, 1$ or 2, S is a point-cone, line-cone or plane-cone, respectively. A non-conical spread may be included by putting $p = -1$.

If S is immersed in F_q, all the spreads of its tree lie in F_q. Its tangent spreads and all the spreads S^{ij} $(j \leqq i)$ are themselves immersed in F_q, while its focal spreads and the other spreads of its tree may be restricted to smaller flats lying in F_q.

Dually, if S is a cone whose vertex is F_p, all the spreads of its tree are conical and enclose F_p. Its focal spreads and all the spreads S_{ij} $(j \leqq i)$ have the same vertex F_p, while its tangent spreads and the other spreads of its tree may have larger vertices enclosing F_p.

The Effect of Projection and Section on the Structure of a Spread.

16. Let us consider the effect of projection and section on the structure of a spread. Two spreads, or the primary trees determined by them, will have the same *structure*, let us say, if these trees consist of the same number of spreads, with the same arrangement and distribution of ascending and descending series, and with equal ranges for corresponding spreads. The dimensionalities of corresponding spreads must, therefore, either be equal or differ by a constant.

Let S be an m-spread immersed in F_q; if S is conical, let its vertex be F_p.

If we project S from a fixed point P, the projecting spread V may be regarded as the common enclosing spread of S and $P = S^0_1$.

The spread of intersection U of S and a fixed hyperplane* Q may be regarded as the common enclosed spread of S and $Q = S^0_{n-1}$.

We first consider two extreme cases.

If S is a restricted spread and the point P lies outside of its enclosing flat F_q, V is a conical $(m+1)$-spread of the same structure as S. Its vertex is the connecting flat of P and F_p; it (V) is immersed in the connecting flat of P and F_q.

If S is a conical spread and the hyperplane Q does not enclose its vertex F_p, U is a restricted $(m-1)$-spread of the same structure as S. It is immersed in the flat of intersection of Q and F_q; its vertex (when it is conical) is the flat of intersection of Q and F_p.

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* A *hyperplane* is an $(n-2)$-flat.

This case includes the operations of projection and section in the ordinary or narrow sense of the words.

If S is conical and P lies in or coincides with F_p, V coincides with S.

If S is restricted and Q passes through or coincides with F_q, U coincides with S.

This trivial case is included merely for the sake of completeness. These are obviously the only two cases in which the structure of a spread is unaltered by projection from a point or section by a hyperplane.

17. In every other case

P is enclosed in F_q but not in F_p. P is then not enclosed in S, but must be enclosed in one of its tangent spreads S^{i+1}. This can happen in two ways. Either S^{i+1} is the highest tangent spread of the tree, of range zero, and coincides with F_q; or S^{i+1} is conical, with a vertex $F_{p'}$ $(p' > p)$ enclosing F_p, and P lies in $F_{p'}$ (but outside of F_p).

Q encloses F_p but not F_q. Q does not then enclose S, but must enclose one of its focal spreads S_{i+1}. This can happen in two ways. Either S_{i+1} is the lowest focal spread of the tree, of range zero, and coincides with F_p; or S_{i+1} is restricted to a flat $F_{q'}$ $(q' < q)$ enclosed in F_q, and Q encloses $F_{q'}$ (without enclosing F_q).

18. Let the point P be enclosed in S^{i+1}, in either one of these two ways, but not in S^i, and therefore not in $S^{i-1}, \ldots, S$. The range r_{i+1} of S^{i+1} must then be less than the range r_i of S^i. By the right-hand theorem of § 5, V^j is the common enclosing spread of S^j and P, for all values of j. Hence, for $j > i$, $V^j = S^j$; and for $j \leqq i$, V^j is of dimensionality one unit higher than S^j. The range of V^i is $r_i - 1$, which is less than that of V^{i-1}. Consequently, the spread $V^{i\beta\gamma\delta\cdots\cdot}$, although enclosing the common enclosing spread of $S^{i\beta\gamma\delta\cdots\cdot}$ and P,* does not, in general, coincide with it; and this part of the tree determined by V has a different structure from the corresponding part of the tree determined by S.

But it is not difficult to see that every spread $V^{\alpha\beta}$ $(0 \leqq \alpha < i)$ coincides with the common enclosing spread of $S^{\alpha\beta}$ and P, has the same range as $S^{\alpha\beta}$, and has a dimensionality one unit higher. The same relation also holds between their tangent spreads $V^{\alpha\beta\gamma}$ and $S^{\alpha\beta\gamma}$, provided $S^{\alpha\beta\gamma}$ does not enclose P. There exists, however, a value j of γ, such that P is enclosed in $S^{\alpha\beta, j+1}$, but not in $S^{\alpha\beta j}$. Hence, $S^{\alpha\beta, j+1}$ can not be enclosed in S^i; and "P. D. G.," § 32, shows that $\alpha - \beta + j \geqq i$.

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* See §§ 3–6.

The same line of argument before applied to S^{i+1} and S^i can now be applied here. Thus, if $\gamma > j$, $V^{\alpha\beta\gamma} = S^{\alpha\beta\gamma}$; if $\gamma = j$, $V^{\alpha\beta\gamma}$ is of range one unit less, and dimensionality one unit greater, than $S^{\alpha\beta\gamma}$; and if $\gamma < j$, $V^{\alpha\beta\gamma}$ is of the same range as $S^{\alpha\beta\gamma}$ and dimensionality one unit greater. Starting with the latter spreads, in which $\gamma < j$, and taking their focal spreads $V^{\alpha\beta\gamma\delta}$, we can continue in this way, until we have exhausted the spreads of the entire tree.

We see, then, that the primary tree determined by the projecting spread V consists, in general, of *three essentially different parts*: (1) a part which coincides with the corresponding part of the tree determined by S; (2) a part which has the same structure as the corresponding part of the tree determined by S, but in which the dimensionalities are one unit higher; and (3) a part which has a radically different structure from the corresponding part of the tree determined by S. Part (1) consists of the partial trees determined by V^{i+1}, the various spreads $V^{\alpha\beta, j+1}$ $(\alpha < i)$, etc. Part (2) consists of those spreads $V^{\alpha\beta\gamma \cdots}$ for which $\alpha < i$, $\gamma < j$, etc. Part (3) consists of those portions of the trees determined by V^i, the various spreads $V^{\alpha\beta j}$ $(\alpha < i)$, etc., which are not included in part (1).

19. It may happen, however, that one or more subdivisions of part (3) coincide with portions of the tree determined by S, though not with the corresponding portions. For instance, if $r_{i+1} = r_i - 1$, V^i has the same range as its tangent spread $V^{i+1} = S^{i+1}$, and so coincides with the focal spread of the latter. Similarly, if the range of $S^{\alpha\beta, j+1}$ is one unit less than that of $S^{\alpha\beta j}$, $V^{\alpha\beta j} = S^{\alpha\beta, j+1, 1}$. If this holds for all the subdivisions of part (3), the entire tree determined by V has the same structure as that portion of the tree determined by S which remains after the excision of S^i, the various spreads $S^{\alpha\beta j}$, etc., and certain spreads determined by them.

Part (2) will evidently be absent, if, and only if, $i = 0$; in this case all the subdivisions of parts (1) and (3) except the first will also be absent.

20. Of the correlative discussion we shall merely state the principal results. Let the hyperplane Q enclose S_{i+1}, but not $S, \ldots,$ or S_i; also let Q enclose $S_{\alpha\beta, j+1}$ $(\alpha < i)$, but not $S_{\alpha\beta j}$; then $\alpha - \beta + j \geqq i$.

The tree determined by the spread of intersection U of S and Q consists of three parts. Part (1) coincides with the corresponding part of the tree determined by S, and is made up of the partial trees determined by U_{i+1}, the spreads $U_{\alpha\beta, j+1}$ $(\alpha < i)$, etc. Part (2), consisting of those spreads $U_{\alpha\beta\gamma \ldots}$ for which $\alpha < i$, $\gamma < j$, etc., has the same structure as the corresponding part of the tree determined by S, but the dimensionalities of its spreads are one unit lower. Part (3), consisting of certain portions of the trees determined by U_i, the spreads $U_{\alpha\beta j}(\alpha < i)$, etc., has a different structure from the corresponding part of the tree determined by S.

21. The following pairs of correlative theorems are easily deduced from what we have said.

If a spread S is projected from a point P by a spread V, and if in the tree determined by S every spread not enclosing P, whose tangent spread does enclose P, has a range exactly one unit greater than its tangent spread, then and only then does the tree determined by V have the same structure as a part, or the whole, of the tree determined by S.

If U is the spread of intersection of a given spread S and a hyperplane Q, and if in the tree determined by S every spread not enclosed in Q, whose focal spread is enclosed in Q, has a range exactly one unit greater than its focal spread, then and only then does the tree determined by U have the same structure as a part, or the whole, of the tree determined by S.

22. If P is enclosed in the immersing flat F_q of S, but not in any of the other spreads of the tree determined by S, then part (1) of the tree determined by V will reduce to the single flat F_q, and conversely.

If Q encloses the vertex F_p of S (when S is conical), but does not enclose any of the other spreads of the tree determined by S, then part (1) of the tree determined by U will reduce to the single flat F_p, and conversely.

If P *is enclosed in* F_q *but not in any of the other spreads of the tree determined by* S*, and if every spread of this tree whose tangent spread is* F_q *is of range* 1*, then and only then does the entire tree determined by* V *have the same structure as a part of the tree determined by* S*, the dimensionalities of all its spreads being one unit higher, with the single exception of the immersing flat, which is* F_q.

If Q *encloses* F_p *but does not enclose any of the other spreads of the tree determined by* S*, and if every spread of this tree whose focal spread is* F_p *is of range* 1*, then and only then does the entire tree determined by* U *have the same structure as a part of the tree determined by* S*, the dimensionalities of all its spreads being one unit lower, with the single exception of the vertex, which is* F_p.

23. If P is enclosed in S^1 but not in S, the tree determined by V consists of two parts, of which the first, determined by V^1, coincides with the corresponding part determined by S^1, and the second, determined by V, has a different structure from the corresponding part determined by S.

If Q encloses S_1 but not S, the tree determined by U consists of two parts, of which the first, determined by U_1, coincides with the corresponding part determined by S_1., and the second, determined by U, has a different structure from the corresponding part determined by S.

A necessary and sufficient condition that the entire tree determined by V coincide with a part of the tree determined by S, is that P be enclosed in S^1 but not in S, and that the range of S^1 be one unit less than that of S.

A necessary and sufficient condition that the entire tree determined by U coincide with a part of the tree determined by S, is that Q enclose S_1 but not S, and that the range of S_1 be one unit less than that of S.

IV. ENUMERATION OF ARBITRARY FUNCTIONS.

24. We shall now take up certain questions concerning the degree of generality of m-spreads generated by ∞^1 flats in F_{n-1}; this is determined by counting the number of independent and essential functions of the parameter ω which enter into their equations.

For instance, the general curve or one-spread in F_{n-1}, whose equations are

$$X_j = A_j(\omega), \quad j = i, \ldots, n,$$

depends apparently on n functions; but by means of the transformations (a) and (c), of "P. D. G.," § 2, two of these functions can be reduced to the values 1 and ω, respectively; the other $n-2$ functions are independent and essential. Using the notation of Kasner,* we may say that the number of curves in F_{n-1} is $\infty^{(n-2)f_1}$.

Similarly, the general m-spread

$$X_j = \sum_{i=1}^{m} a_i A_{ij}(\omega), \quad j = 1, \ldots, n,$$

involves the mn functions $A_{ij}(\omega)$; but m^2 of these functions can be reduced to constants by the transformation (b), of "P. D. G.," § 2, and one other function can be chosen as a new parameter ω by means of (a), of the same section; so we see that *the general m-spread S_m in F_{n-1} depends essentially on $m(n-m)-1$ arbitrary functions.* By arbitrary functions we mean functions which are continuous and possess a certain finite number of successive derivatives, but otherwise are arbitrary.

Geometrically, we obtain the same result by making use of the well-known† fact that the choice of every generating $(m-1)$-flat of S_m depends on $m(n-m)$ arbitrary constants; hence, a continuous system of ∞^1 such generators depends on $m(n-m)$ arbitrary functions of ω, one of which, again, can be chosen to replace ω.

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* *Bulletin of the American Mathematical Society*, Vol. XIX (October, 1912), p. 14.

† Cf. Schoute, "Mehrdimensionale Geometrie," erster Teil (1902), p. 253.

The range of the general m-spread in F_{n-1} is obviously the smaller of the two integers m and $n-m$. If n is even, the number $m(n-m)-1$ will be a maximum, when $m=\frac{n}{2}$; and if n is odd, it will be a maximum, when $m=\frac{n-1}{2}$ or $\frac{n+1}{2}$. Hence, the most general spread in F_{n-1}, if n is even, is $S^{n/2}_{n/2}$, which depends on $\frac{n^2-4}{4}$ arbitrary functions; and if n is odd, it is either $S^{(n-1)/2}_{(n-1)/2}$ or $S^{(n-1)/2}_{(n+1)/2}$, each of which depends on $\frac{n^2-5}{4}$ arbitrary functions.

25. If a spread S is of higher range than both its tangent spread S^1 and its focal spread S_1, it remains to a certain extent arbitrary, even when S^1 and S_1 are known; that is, when all the spreads of its primary tree except itself are known. For, putting $S=S^r_m$, $S^{1,0}=S^{r_1}_{m+r}$, $S_{1,0}=S^{r_{-1}}_{m-r}$,* we see that S can be chosen to be any spread enclosed in $S^{1,1}=S_{m+r-r_1}$* and enclosing $S_{1,1}=S_{m-r+r_{-1}}$, provided that the common enclosing spread of S and $S^{1,2}$ be precisely $S^{1,1}$ and not a lower spread enclosed in it, and provided that the common enclosed spread of S and $S_{1,2}$ be precisely $S_{1,1}$ and not a higher spread enclosing it. These provisos, however, are in general satisfied. In other words, the generators of S can in general be chosen to be any $(m-1)$-flats lying in the corresponding $(m+r-r_1-1)$-flats of $S^{1,1}$ and passing through the corresponding $(m-r+r_{-1}-1)$-flats of $S_{1,1}$.

Hence † the choice of each generator of S depends on $(r-r_1)(r-r_{-1})$ arbitrary constants. That is, *if there exists* ‡ *a spread of range* r, *having a given tangent spread of range* r_1 *and a given focal spread of range* r_{-1}, *there exist exactly* $\infty^{(r-r_1)(r-r_{-1})f_1}$ *such spreads.*

26. In a similar manner we can easily prove the following dual theorems, which may be regarded as an extension of "P. D. G.," § 51.

If a spread S^r_m, *in* F_{n-1}, *has a given tangent spread of range* r_1, *the determination of* S^r_m *depends on* $(r-r_1)m$ *arbitrary functions.*	*If a spread* S^r_m, *in* F_{n-1}, *has a given focal spread of range* r_{-1}, *the determination of* S^r_m *depends on* $(r-r_{-1})(n-m)$ *arbitrary functions.*

27. Let us now determine the degree of generality of S^r_m *per se*; that is, when neither its tangent spread nor its focal spread is known. In the first place, the general case is the one in which the ascending and descending series determined by S^r_m are both regular so far as the given values of m, r and n

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* For the notation, see "P. D. G.," § 17. † Cf. Schoute, *loc. cit.* ‡ See "P. D. G.," §§ 51, 52.

will allow. That is, the spreads of the ascending (descending) series are all of range r except the last one, which is of range zero, and the next to the last one, which may be of range $< r$. For, any additional irregularity is due to the existence of additional linear relations which diminish the generality of the spread. Moreover, the different spreads of a regular series all have the same degree of generality.

Let $m = rk + r_1$, where k is an integer and $0 \leq r_1 < r$. The $(k-1)$-st focal spread of S^r_m will be an $S^r_{r+r_1}$, in the general case, and the k-th focal spread an $S^{r_1}_{r_1}$. S^r_m will be uniquely determined by first choosing $S^{r_1}_{r_1}$, which, by the theorem of § 24, depends on $r_1(n-r_1)-1$ arbitrary functions, and then choosing $S^r_{r+r_1}$, which, by the right-hand theorem of § 26, depends on $(r-r_1)(n-r-r_1)$ additional arbitrary functions. Hence, the total number of arbitrary functions, all independent, on which S^r_m depends, is $r(n-r)-1$.

That is, *the general m-spread of range r in F_{n-1} is determined by $r(n-r)-1$ arbitrary functions.* Since this number is independent of r_1 and of m, it follows that the spreads S^r_r, S^r_{r+1}, ..., S^r_{n-r} all have the same degree of generality.

28. We proceed to the consideration of spreads whose ascending or descending series is irregular. We first take the case where the descending series is irregular, and the ascending series regular (so far as possible).

THEOREM. *In F_{n-1} the most general spread S^r_m $(r = r_0)$, whose successive focal spreads are of ranges r_{-1}, r_{-2}, ..., $r_{-(t-1)}$, 0, whose t-th focal spread is a flat $S^0_{m_{-t}}$, and whose tangent spreads are entirely arbitrary, is determined by μ arbitrary functions, where*

$$\mu = r(n-m_{-t}-r) - \sum_{i=-1}^{-(t-2)} r_{i-1}(r-r_i) - 1 = r(n-m) + \sum_{i=0}^{-(t-2)} r_i r_{i-1} - 1. \qquad (9)$$

As to the proof, it is sufficient to remark that $S^r_m = S$ can be determined by first choosing $S_{t,0}$, then $S_{t-1,0}$, etc., and finally S itself. Since $S_{t,0}$ is a flat S^0_{-t}, its choice depends on $m_{-t}(n-m_{-t})$ arbitrary constants, and not on any arbitrary functions. When $S_{i+1,0}$ has been chosen, $S_{i,0}$, of dimensionality $m - \sum_{a=0}^{-(i-1)} r_a$, is restricted only by the fact that it must enclose $S_{i+1,1}$ and be enclosed in F_{n-1}; hence, if $t-1 > i > 0$, its choice depends on

$$(r_{-i} - r_{-(i+1)})\,(n - m + \sum_{a=0}^{-(i-1)} r_a)$$

arbitrary functions. When so chosen, it will, in general, have a regular series of tangent spreads S_{ij} $(j = 1, \ldots, i)$, and will therefore satisfy the condition of "P. D. G.," § 52.

29. Dually, it is clear that *the most general spread S_m^r $(r = r_0)$, whose successive tangent spreads are of ranges $r_1, r_2, \ldots, r_{u-1}$, 0, whose u-th tangent spread is a flat $S_{m_u}^0$, and whose focal spreads are entirely arbitrary, is determined by μ' arbitrary functions, where*

$$\mu' = r(m_u - r) - \sum_{i=1}^{u-2} r_{i+1}(r - r_i) - 1 = rm + \sum_{i=0}^{u-2} r_i r_{i+1} - 1. \tag{9'}$$

30. We next proceed to prove that *if there exists a spread of range r, whose first tangent and focal spreads are of ranges r_1 and r_{-1}, respectively, and whose second tangent and focal spreads are given spreads of ranges r_2 and r_{-2}, respectively, then there exist just ∞^{kf_1} such spreads, where*

$$\begin{aligned} k &= (r-r_1)(r-r_{-1}) + 2(r-r_1)(r_{-1}-r_{-2}) + 2(r-r_{-1})(r_1-r_2) + 3(r_1-r_2)(r_{-1}-r_{-2}) \\ &= r^2 + r(r_1+r_{-1}) - 2r(r_2+r_{-2}) - (r_1 r_{-2} + r_2 r_{-1}) + 3 r_2 r_{-2}. \end{aligned} \tag{10}$$

Proof. The required spread $S = S_m^r$ can be determined by first choosing $S^{1,0}$, then $S_{1,0}$, and finally S itself. $S^{1,0}$ is the most general spread S_{m+r} enclosed in $S^{2,1} = S_{m+r+r_1-r_2}$ and enclosing $S_{2,3} = S_{m-r-r_{-1}+3r_{-2}}$, and so depends on $(r_1 - r_2)(2r + r_{-1} - 3r_{-2})$ arbitrary functions. Then $S_{1,0}$ is the most general spread S_{m-r} enclosing $S_{2,1} = S_{m-r-r_{-1}+r_{-2}}$ and enclosed in $S^{1\,2} = S_{m+r-2r_1}$, and so depends on $(r_{-1} - r_{-2})(2r - 2r_1)$ additional functions. Finally, S, by the theorem of § 25, depends on $(r - r_1)(r - r_{-1})$ additional functions. The sum of these numbers gives (10), and proves the theorem.

31. We are now prepared to take up the case where the ascending and descending series are both irregular.

Theorem. *In F_{n-1} the most general spread S_m^r $(r = r_0)$, whose successive tangent spreads are of ranges $r_1, r_2, \ldots, r_{u-1}$, 0, the last one being an $S_{m_u}^0$, and whose successive focal spreads are of ranges $r_{-1}, r_{-2}, \ldots, r_{-(t-1)}$, 0, the last one being an $S_{m_{-t}}^0$, is determined by λ arbitrary functions, where*

$$\begin{aligned} \lambda &= r(m_u - m_{-t} - r) - \sum_{i=1}^{u-2} r_{i+1}(r - r_i) - \sum_{i=-1}^{-(t-2)} r_{i-1}(r - r_i) - 1 \\ &= r^2 + \sum_{i=0}^{u-2} r_i r_{i+1} + \sum_{i=0}^{-(t-2)} r_i r_{i-1} - 1. \end{aligned} \tag{11}$$

32. The method of proof will be sufficiently clear, if carried out for the special case in which $t = u = 3$, $m_{-t} = 0$, $m_u = n$; so that

$$n = 2r + (r_1 + r_2) + (r_{-1} + r_{-2}), \tag{12}$$

where

$$r \geqq r_1 \geqq r_2 \text{ and } r \overset{=}{>} r_{-1} \overset{=}{>} r_{-2}. \tag{13}$$

We first find the number of arbitrary functions that determine $S_2 = S^{r_{-2}}_{r_{-2}}$ and $S^2 = S^{r_2}_{n-r_2}$. The number of additional functions that determine S^1, S_1 and S will then be given by (10), § 30. When S_2 is chosen in the most general manner possible for the given values of the ranges, its fourth tangent spread $S_{2,4}$, which is enclosed in S^2, will be of dimensionality $5\,r_{-2}$, provided

$$n \geqq r_2 + 5\,r_{-2}. \tag{14}$$

In this case S_2 depends on $r_{-2}(n-r_{-2})-1$ functions, and S^2 on $r_2(n-r_2-5r_{-2})$ additional functions; so that together they depend on

$$r_2(n-r_2)+r_{-2}(n-r_{-2})-5\,r_2\,r_{-2}-1=(2\,r+r_1+r_{-1})\,(r_2+r_{-2})-3\,r_2\,r_{-2}-1 \tag{15}$$

arbitrary functions.

If (14) is not satisfied, this method fails; but we can proceed to choose S^2 first and then S_2. For, $S^{2,4}$ encloses S_2 and can be chosen to be of dimensionality $n-5\,r_2$, provided

$$n \geqq r_{-2} + 5\,r_2. \tag{14$'$}$$

In this case S^2 depends on $r_2(n-r_2)-1$ functions, and S_2 on $r_{-2}(n-r_{-2}-5\,r_2)$ additional functions; so that the number of functions on which both depend is given by the formula (15), as before. But one of the two conditions (14), (14′) must in every case be satisfied. For if not, we should have, by addition,

$$n < 3\,(r_2 + r_{-2}),$$

which would contradict (12) and (13).

Finally, by (15) and (10), we find that the total number of arbitrary functions on which the original spread S depends is

$$r(n-r)-r_2(r-r_1)-r_{-2}(r-r_{-1})-1 = r^2+(r\,r_1+r_1\,r_2)+(r\,r_{-1}+r_{-1}\,r_{-2})-1,$$

which proves formula (11) for the special case under consideration.

It is clear from (11) that the maximum value of λ for given values of n and r is $r\,(n-r)-1$, and that this value is obtained by putting $m_u = n$, $m_{-t} = 0$ and $r_{u-2} = \ldots\ldots = r_1 = r = r_{-1} = \ldots\ldots = r_{-(t-2)}$. This verifies the theorem of § 27 from another point of view.

V. CLASSIFICATION.

33. In classifying spreads generated by ∞^1 flats from the projective differential standpoint, we shall employ three different kinds of classification: first, a broad classification, depending only on the structure of the two principal

series, one ascending and the other descending, determined by the given spread; second, a narrower and more detailed classification, taking account of the branch series as well, and depending on the structure of the entire primary tree determined by the spread; and third, a still more detailed classification, taking account of the common enclosed and enclosing spreads of the various spreads of the tree, and depending, therefore, on the structure of the extended tree determined by the spread.

If two spreads S_m and $T_{m'}$, of different dimensionalities $(m < m')$, have the same structure or belong to the same class according to any one of these three principles of classification, it is evident that by means of one or more projections or sections of the ordinary kind (see § 16), we can obtain from either one of the spreads a new spread of the same dimensionality and class as the other. Thus, if $m' = m + \mu$, then by μ successive projections we obtain from S_m a spread $S_{m'}$ of the same class as $T_{m'}$, and by μ successive sections we obtain from $T_{m'}$ a spread T_m of the same class as S_m.

34. For instance, in four-dimensional space F_4 all plane curves S_1^1 (not including straight lines), all cones T_2^1 immersed in three-flats, and all line-cones U_3^1 immersed in F_4, belong to the same class K_1. Their respective trees are

$$\left.\begin{array}{ccc} S_3^0 & T_4^0 & U_5^0 \\ S_2^1 & T_3^1 & U_4^1 \\ S_1^1 & T_2^1 & U_3^1 \\ S_0^0 & T_1^0 & U_2^0 \end{array}\right\}$$

where the order from the bottom up is in each case: the first focal spread, the given spread, the first tangent spread and the second tangent spread. T_1^0 is the point-vertex of T_2^1, T_3^1 is the system of tangent planes to T_2^1, T_4^0 is the three-flat in which T_2^1 is immersed, etc. Every S_1^1 projects into a T_2^1, and every T_2^1 into a U_3^1. Moreover, the spreads S_2^1, T_3^1 and U_4^1 belong to a class K_2 different from K_1; for although the tree determined by S_2^1 is the same as that determined by S_1^1, the corresponding spreads are not of the same range.

Since the principle of duality holds between the pairs of spreads S_1^1 and U_4^1, S_2^1 and U_3^1, T_2^1 and T_3^1, we can speak of K_1 and K_2 as dual, or correlative, classes of spreads.

The ranges 0, 1, 1, 0 of the spreads of these trees serve to characterize the classes K_1 and K_2; hence, we write $K_1 = (0\bar{1}10)$ and $K_2 = (01\bar{1}0)$, indicating the range of the original spread by a bar. The symbol (0110), without the bar, will then denote the two classes taken together.

Developable Spreads.

35. Returning to the general discussion, we shall now consider systematically the different classes of spreads existing in F_{n-1}, in the order of their simplicity.

First of all, there is the trivial class (0), consisting of the spreads of range zero, or the fixed points, lines, planes, etc.

Then come the various classes (011....110) of spreads of range 1, or *developable spreads.* Since their trees have no branch series, the three methods of classification here yield precisely the same results. The simplest class of developables is $(0\bar{1}0)$, which includes the ranges of points, the pencils of lines, the pencils of planes, etc. Notice that the straight line considered as generated by its points belongs here, while considered as a whole it is of range 0. Next in order are the two classes (0110) considered (for $n=5$) in the preceding section. After them come the three classes (01110), of which the simplest (non-conical) representatives are: for $(0\bar{1}110)$, a curve immersed in a three-flat; for $(01\bar{1}10)$, its tangent surface; and for $(011\bar{1}0)$, its system of osculating planes. The classes $(0\bar{1}110)$ and $(011\bar{1}0)$ are dual, while $(01\bar{1}10)$ is self-dual.

Obviously, the total number of classes of developables in F_{n-1} is $1+2+\dots+(n-1)=\frac{1}{2}n(n-1)$, of which $n-1$ are classes of *proper* spreads in F_{n-1}, in the sense that they have no representatives in a space of lower dimensionality than $n-1$. A proper spread, therefore, is a *non-conical* spread *immersed* in F_{n-1}.

Spreads of Range 2.

36. Spreads of range 2 are of four general types,

$$(0,2\dots.2,0),\quad (0,2\dots.2,1\dots.1,0),\quad (0,1\dots.1,2\dots.2,0),$$
$$\text{and}\quad (0,1\dots.1,2\dots.2,1\dots.1,0).$$

Those of the first type may be called *regular* spreads, since the entire tree, exclusive of the first and last spreads, consists of a single regular series. All developable spreads are, therefore, also regular. Those of the second type possess a descending branch series, of range 1, not indicated in the symbol; similarly, those of the third type have an ascending branch series, and those of the fourth type have two branch series, one ascending and the other descending.

The simplest class of regular spreads of range 2 is $(0\bar{2}0)$, which is represented in F_3 by a skew ruled surface S_2^2. If n is even, there are evidently $\frac{1}{8}n(n-2)$ classes of regular spreads of range 2 in F_{n-1}, of which $\frac{1}{2}(n-2)$

are proper; if n is odd, there are $\frac{1}{8}(n-1)(n-3)$ classes, but not one is proper.

37. When we come to the non-regular spreads, the number of classes depends on the particular method of classification. We shall for the present use only the first and broadest method, reserving a minuter classification of a few of the simpler spreads for sections 42–73.

In the case of the second type of spreads of range 2, the total number of classes in F_{n-1}, if n is even, is easily computed to be $\frac{1}{24}n(n-2)(n-4)$; and the number of proper classes is $\frac{1}{8}(n-2)(n-4)$. If n is odd, the total number of classes is $\frac{1}{24}(n-1)(n-2)(n-3)$; and the number of proper classes is $\frac{1}{8}(n-1)(n-3)$. The same figures hold for the third type.

In the case of the fourth and last type, the total number of classes in F_{n-1}, if n is even, is $\frac{1}{96}n(n-2)(n-4)^2$; and $\frac{1}{24}(n-2)(n-3)(n-4)$ of them are proper classes. The total number, if n is odd, is $\frac{1}{96}(n-1)^2(n-3)(n-5)$; and $\frac{1}{24}(n-1)(n-3)(n-5)$ of them are proper.

Spreads of Range Higher than 2.

38. As the range increases, the number and complexity of the different types increases rapidly. For instance, if the range of S is 3, the principal ascending series determined by S may include spreads of ranges 1 or 2, or both or neither, and the same is true of the principal descending series. Hence, there are $4 \cdot 4 = 16$ types, of which the simplest is the regular type

$$(0,3\ldots.3,0),$$

and the most complicated is

$$(0,1\ldots.1,2\ldots.2,3\ldots.3,2\ldots.2,1\ldots.1,0).$$

Spreads of the latter type have two principal branch series of range 1, ascending and descending respectively, and two principal branch series of range 2, ascending and descending; moreover, each of the latter may determine a secondary branch series of range 1 and possibly a connecting series.

The simplest class, however, in which a connecting series exists, is obviously $(02\overline{3}20)$; and it is represented in F_9 by a spread

$$S_5^3 = [A_1, A_2, A_1^1, A_2^1, A_3^1],$$

whose fundamental directrices satisfy a differential equation of the form*

$$\sum_{\beta=0}^{3} \sum_{\lambda=1}^{2} h_{\beta\lambda} A_\lambda^{3-\beta} = 0.$$

Proper Spreads in F_4 and in F_5.

39. From this point on we shall confine our attention to four and five dimensions, and to the classes of proper, non-developable spreads.

In F_4 there are just two such classes, $(0\bar{2}10)$ and $(01\bar{2}0)$, and they are correlatives. In view of the theorem of § 27, each of these classes contains ∞^{5f_1} spreads. The first is the class of skew† ruled surfaces S_2^2 immersed in F_4, and the second is the class of non-conical skew planar‡ spreads S_3^2. The fact that the first tangent spread of S_2^2 is an S_4^1, and that the second tangent spread is $S_5^0 = F_4$, may be expressed by saying that two consecutive generating lines of S_2^2 are connected by a three-flat and that the connecting flat of three consecutive generators is F_4 itself. Dually, we may say that two consecutive generating planes of S_3^2 meet in a point, and three consecutive generators do not meet.

40. In F_5 there are six classes of proper, non-developable spreads:

$(0\bar{2}20)$, $(02\bar{2}0)$, $(0\bar{2}110)$, $(011\bar{2}0)$, $(01\bar{2}10)$ and $(0\bar{3}0)$,

of which the first two and the last are regular.

The class $(0\bar{2}20)$ consists of ∞^{7f_1} spreads, namely the skew ruled surfaces S_2^2 whose tangent spreads are of range 2; these tangent spreads S_4^2 constitute the dual class $(02\bar{2}0)$; their focal spreads are then the original ruled surfaces S_2^2. In other words, two consecutive generating lines of S_2^2 are connected by a generating three-flat of S_4^2; and two consecutive generating three-flats of S_4^2 intersect in a generating line of S_2^2. Three consecutive generators of S_2^2 are connected by F_5, and three consecutive generators of S_4^2 do not meet.

On the other hand, the class $(0\bar{2}110)$ consists of the skew ruled surfaces S_2^2 immersed in F_5, whose tangent four-spreads are developable; two consecutive generators of S_2^2 are connected, as in the preceding case, by a three-flat, but three consecutive generators are now connected by a four-flat. The dual class $(011\bar{2}0)$ consists of the non-conical skew four-spreads S_4^2 whose focal spreads are developable surfaces. Two consecutive generators of S_4^2 meet in a line and three consecutive generators meet in a point. Each of these classes, in view of §§ 28, 29, contains ∞^{6f_1} spreads.

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* See "P. D. G.," §§ 56 and 69.

† A non-developable spread may be called a *skew* spread.

‡ A three-spread S_3, being generated by ∞^1 planes, may be called a *planar* spread.

The self-dual class $(0\bar{3}0)$ includes the planar three-spreads of range 3, S_3^3, the number of which is ∞^{8f_1}. They have no tangent or focal spreads, properly speaking; that is, two consecutive generating planes of S_3^3 do not meet, and their connecting flat is F_5.

Finally, the self-dual class $(01\bar{2}10)$ includes the proper planar three-spreads of range 2, S_3^2, the number of which is ∞^{7f_1} (cf. §§ 27 and 31). Their tangent five-spreads and focal curves are necessarily developable. Two consecutive generating planes of S_3^2 meet in a point and are connected by a four-flat; three consecutive generators do not meet and are connected by F_5.

41. The difference between planar three-spreads of ranges 3, 2 and 1 in F_5 may be expressed by saying that two *consecutive* planes of S_3^3 do not meet, those of S_3^2 meet in a point, and those of S_3^1 meet in a line. This must be understood to mean merely that the focal spreads in the three cases are of dimensionalities 0, 1 and 2, respectively. Two *neighboring* generators of S_3^3 and of S_3^1 will not, in general, meet at all. The relative position of two neighboring generators can be determined by introducing the metric concepts of distance and angle, and recalling that two planes in F_5 have a distance α, a maximum angle β and a minimum angle γ.* Then it can easily be shown that in the case of two neighboring planes of S_3^3 the three quantities α, β, γ are infinitesimals of the same order; in the case of S_3^2 two of them are infinitesimals of the same order, while the third is of higher order; and in the case of S_3^1 one of them is of lower order than the other two.

In sub-classifying these spreads in F_4 and F_5 with respect to the nature of their branch series and extended trees, only the irregular spreads need be taken into account. Thus we have two classes to consider in F_4 and three classes in F_5.

Skew Ruled Surfaces Immersed in F_4.

42. Let $S = S_2^2$ be a skew ruled surface immersed in F_4. Its homogeneous parametric equations are

$$X_j = a_1 A_{1j}(\omega) + a_2 A_{2j}(\omega),$$

where j runs from 1 to 5, and the parameters are ω and $a_1 : a_2$. Using the notation and methods of "P. D. G.," §§ 29, 39–45, 55–62 and 69,† we write

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* Cf. Schoute, *loc. cit.*, pp. 53–72.

† According to the more usual custom, however, we denote the second derivative of A_1 with respect to ω by A_1'' instead of A_1^2 (as in "P. D. G.") and reserve the latter expression for the square of A_1.

$$S = S_2^2 = [A_1, A_2],$$

$$S^{1,0} = S_4^1 = [A_1, A_2, A_1', A_2'],$$

and

$$S^{2,0} = S_5^0 = F_4 = [A_1, A_2, A_1', A_2', A_1'', A_2''].$$

The elements of the array

$$\left.\begin{array}{cc} A_1''' & A_2''' \\ A_1'' & A_2'' \\ A_1' & A_2' \\ A_1 & A_2 \end{array}\right\}$$

are connected by a linear homogeneous relation

$$\Sigma g_{0i} A_i''' + \Sigma g_{1i} A_i'' + \Sigma g_{2i} A_i' + \Sigma g_{3i} A_i = 0, \tag{16}$$

and the elements of the three bottom rows by a relation

$$\Sigma h_{0i} A_i'' + \Sigma h_{1i} A_i' + \Sigma h_{2i} A_i = 0, \tag{16'}$$

where

$$\begin{vmatrix} g_{01} & g_{02} \\ h_{01} & h_{02} \end{vmatrix} \neq 0.$$

All the quantities involved in these equations are functions of ω.

By means of a transformation to new directrices, of the form (b), § 2, "P. D. G.," together with a transformation of the coefficients, of the form (37) and (38), § 57, "P. D. G.," the differential equations (16) and (16′) can easily be reduced to the canonical form

$$A_2''' + (k_1 A_2)'' + \Sigma (k_{2i} A_i)' + \Sigma k_{3i} A_i = 0 \tag{17}$$

and

$$A_1'' + (l_1 A_2)' + l_2 A_2 = 0. \tag{17'}$$

In all that follows, therefore, we shall assume (17) and (17′) to be the fundamental differential equations of the spread. It is clear that

$$S^{1,1} = S_3^1 = [A_1, A_2, A_1'] \quad \text{and} \quad S^{1,2} = S_2 = [A_1, A_1' + l_1 A_2].$$

43. $S^{1,1}$ is a planar three-spread, whose generating planes contain the generating lines of S; and it is the only such three-spread that is developable, as can easily be seen, either by direct calculation, or by duality from the theorem of § 53. Hence, *every skew ruled surface S immersed in four-space determines one and only one developable three-spread whose generating planes contain the generating lines of S, namely its special enclosing spread $S^{1,1}$.*

44. The focal spread $S^{1,2}$ of $S^{1,1}$ may be, as we shall see, either a developable surface or a fixed line m; but its generators (or generator) will meet the corresponding generators of S in the points of a curve (or range on m)

$$T = T_1^1 = [A_1],$$

which is therefore the common enclosed spread or curve of intersection of S and $S^{1,2}$.

Let $A_1(\omega)$ denote the point on T whose coordinates are $A_{11}(\omega), \ldots, A_{15}(\omega)$. The tangent plane* to S at the point $A_1(\omega)$ is evidently determined by the three points $A_1(\omega)$, $A_2(\omega)$ and $A_1'(\omega)$. But this plane is precisely the corresponding generator of $S^{1,1}$. Hence, the planes of $S^{1,1}$ touch S, and the locus of their points of contact is T. It follows that on a *skew ruled surface* S *in* F_4 *there exists* (*apart from the generators themselves*) *a unique curve* (*or straight line*) T,† *such that the tangent planes to* S *at the points of* T *form a developable three-spread;* T *is the curve of intersection of* S *and* $S^{1,2}$.

The fact that T turns out to be one of the fundamental directrices $[A_1]$ of S is a consequence of choosing the fundamental differential equations in their canonical form (17), (17′).

We are now prepared to classify ruled surfaces in F_4, by means of the values of the coefficients l_1 and l_2 in (17′), as follows:

Class (a), $l_1 \neq 0$,
Class (b), $l_1 = 0$, $l_2 \neq 0$,
Class (c), $l_1 = l_2 = 0$.

Class (a).

45. This is the general case, and depends on five arbitrary functions. $S^{1,2}$ is of range $1 (= S_2^1)$, and its focal spread $S^{1,3}$ is easily found, by the method of "P. D. G.," §§ 16, 19–21, to be

$$S^{1,3} = S_1 = \left[A_1' + l_1 A_2 - \frac{l_2}{l_1} A_1 \right].$$

$S^{1,3}$ is in general a curve (of range 1), but *will be a fixed point* (of range 0), *if and only if* l_2/l_1 *satisfies the Riccati equation*

$$\frac{d}{d\omega}\left(\frac{l_2}{l_1}\right) + \left(\frac{l_2}{l_1}\right)^2 = 0.$$

That is, $S^{1,1}$ is in general non-conical, but may be a point-cone.

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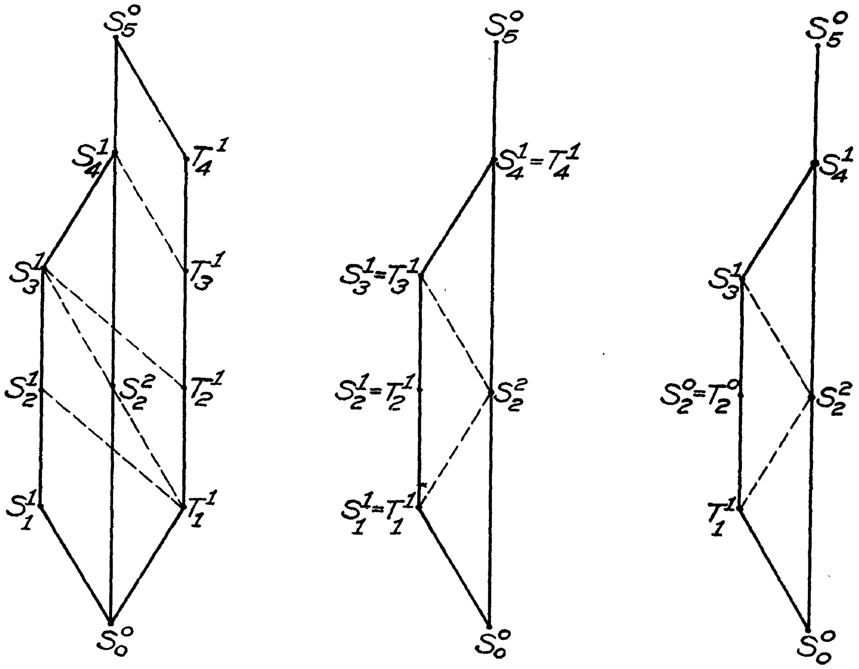
* Note the distinction between the tangent *plane at a point* of S and the tangent *three-flat along a generator* of S; the latter, as defined in "P. D. G.," § 4, is a generator of $S^{1,0}$.

† In F_4, as in F_3, a ruled surface S obviously has a line of striction; but in F_3 there is no purely projective curve on S analogous to T in F_4.

The successive tangent spreads of T are

$$\begin{aligned} T^{1,0} &= T_2^1 = [A_1,\ A_1'], \\ T^{2,0} &= T_3^1 = [A_1,\ A_1',\ (l_1 A_2)' + l_2 A_2], \\ T^{3,0} &= T_4^1 = [A_1,\ A_1',\ (l_1 A_2)' + l_2 A_2,\ (l_1 A_2)'' + (l_2 A_2)']. \end{aligned}$$

T_1^1, T_2^1 and T_3^1 are enclosed in S_2^1, S_3^1 and S_4^1, respectively; but T_2^1 cannot coincide with S_2^1. $T^{3,0}$ is in general a family of ∞^1 three-flats (of range 1), but *will be a fixed three-flat* (of range 0), *if and only if* $(l_1 A_2)''' + (l_2 A_2)''$ *is expressible, by virtue of* (17), *as a linear homogeneous function of the fundamental directrices of* $T^{3,0}$. Hence, T_1^1 is in general a four-space curve, but may be a three-space curve.



CLASS (a) SUB CLASS (a_{11}) CLASS (b) CLASS (c)

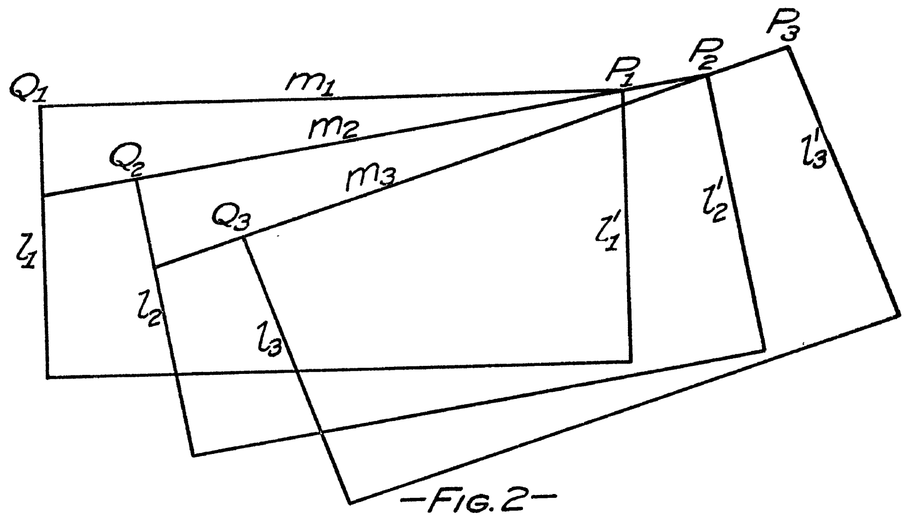
—FIG. 1—

Let r and r' be the ranges of $S^{1,3}$ and $T^{3,0}$, respectively. Their values determine four subclasses into which class (a) is divided, namely:

subclass (a_{11}), $r = 1$, $r' = 1$; subclass (a_{12}), $r = 1$, $r' = 0$;
subclass (a_{21}), $r = 0$, $r' = 1$; subclass (a_{22}), $r = 0$, $r' = 0$.

We have completed the enumeration of the spreads of the extended tree determined by a surface S of class (a). For the subclass (a_{11}) this tree is indicated in Fig. 1, where every spread is connected with its tangent and focal spreads by full lines, vertical or oblique, and with its other enclosed and enclosing spreads by dotted lines.

46. Fig. 2 gives a schematic representation of the surface S^2_2 itself, when it belongs to the subclass (a_{11}) or (a_{12}). The lines l_1, l_2, l_3 are three neighboring generators of S^2_2; the three-flats (l_1, l_2) and (l_2, l_3) intersect in the plane $\pi_2 = (l_2, m_2)$, the limiting position of which is therefore a generator of S^1_3. Let $\pi_1 = (l_1, m_1)$ and $\pi_3 = (l_3, m_3)$ be two other generators of S^1_3; π_1 and π_2 intersect in the line m_2, π_2 and π_3 in the line m_3; m_1, m_2, m_3 may be regarded as three neighboring generators of the developable S^1_2, and P_1, P_2, P_3 as three neighboring points on its edge of regression S^1_1. When S^2_2 belongs to



—Fig. 2—

the subclass (a_{21}) or (a_{22}), the points P_1, P_2, P_3 coincide, and S^1_2 is a cone whose vertex S^0_1 is P_1. Q_1, Q_2 and Q_3, being the points of intersection of l_1 and m_1, l_2 and m_2, l_3 and m_3, respectively, are generators of the curve T^1_1; the tangent planes to S^2_2 at the points Q_1, Q_2, Q_3 are π_1, π_2, π_3. The lines Q_1Q_2 and Q_2Q_3, lying in the planes π_1 and π_2, respectively, may therefore be regarded as generators of T^1_2, the tangent developable of T^1_1; and the plane $Q_1Q_2Q_3$, lying in the three-flat (l_1, l_2), may be regarded as an osculating plane of T^1_1; that is, a generator of T^1_3.

If we start with five consecutive generators of S^2_2, they determine four consecutive three-flats of S^1_4, three consecutive planes of S^1_3, two consecutive lines of S^1_2 and one point of S^1_1.

It is to be observed that of the two fundamental curves T^1_1 and S^1_1, associated with the surface S, the first lies on the surface, while the second does not.

47. In order to obtain the most general skew ruled surface S^2_2 of class (a), we can *first choose at random a developable* S^1_2 *immersed in* F_4, which, if non-

conical, depends on three arbitrary functions, and if conical, on two arbitrary functions. We can *then choose any curve* T_1^1 *lying on* S_2^1, *except its edge of regression* S_1; T_1^1 will be either immersed in F_4 or restricted to an F_3. If it is immersed in F_4, it depends on one additional arbitrary function, but if in F_3, it is simply the curve of intersection of S_2^1 and F_3, and therefore involves no additional arbitrary functions, but merely four arbitrary constants, since F_3 can be any three-flat not passing through the vertex (if any) of S_2^1. Finally, we can *choose* S_2^2 *itself to be any ruled surface whose generators lie in the planes of* S_3^1 (the tangent spread of S_2^1) *and pass through the points of* T_1^1, *with two exceptions, namely,* S_2^1 *itself and the tangent surface* T_2^1 *of* T_1^1; this choice depends on one additional arbitrary function. Consequently, the number ν of arbitrary functions on which S_2^2 depends, for each of the four subclasses of (a), is as follows: $(\mathrm{a}_{11}), \nu = 5$; $(\mathrm{a}_{12}), \nu = 4$; $(\mathrm{a}_{21}), \nu = 4$; $(\mathrm{a}_{22}), \nu = 3$.

Among all the ruled surfaces enclosed in S_3^1 *and enclosing* T_1^1, *just two are developable, namely,* S_2^1 *and* T_2^1, *while all the rest,* ∞^{f_1} *in number, are skew.*

Another method of constructing S_2^2 is to choose T_1^1 first, then S_2^1 as a developable enclosing T_1^1, and finally S_2^2. If S_2^1 is a cone whose vertex is $S_1^0 = F_0$, it is obtained by simply projecting T_1^1 from F_0.

48. As an illustration let us write the equations of a simple rational surface of subclass (a_{22}). Let S_2^1 be a cone whose vertex S_1^0 is $C = (1, 0, 0, 0, 0)$, and whose directrix $T_1^1 = [A_1]$ is a curve immersed in the three-flat $x_1 = 0$. Let the equations of T_1^1 be

$$x_1 : x_2 : x_3 : x_4 : x_5 = 0 : 1 : \omega : \omega^2 : \omega^3;$$

more briefly, we shall write:

$$T_1^1 = [A_1] = (0, 1, \omega, \omega^2, \omega^3).$$

Evidently, $S_2^1 = [C, A_1]$, $S_3^1 = [C, A_1, A_1']$ and $T_2^1 = [A_1, A_1']$, where

$$A_1' = (0, 0, 1, 2\omega, 3\omega^2).$$

The second fundamental directrix A_2 of $S_2^2 = [A_1, A_2]$ can be chosen to be any curve of S_3^1, not situated on S_2^1 or on T_2^1. That is, $[A_2] = [a_1 C + a_2 A_1 + a_3 A_1']$, where a_1, a_2, a_3 are functions of ω or constants, such that $a_1 \neq 0$ and $a_3 \neq 0$. The simplest choice is $a_1 = a_3 = 1, a_2 = 0$, so that $[A_2]$ is the conic

$$[C + A_1'] = (1, 0, 1, 2\omega, 3\omega^2).$$

Then

$$S_2^2 = [A_1, A_2] = (\lambda, 1, \omega + \lambda, \omega^2 + 2\lambda\omega, \omega^3 + 3\lambda\omega^2),$$

where ω and λ are the parameters. By reference to the theorem of § 14, we see that S_2^2 is of order 4, S_2^1 and T_1^1 are of order 3, S_3^1 and T_2^1 are of order 4, and S_4^1 and T_3^1 are of order 3.

Class (b).

49. This case depends on four arbitrary functions. Evidently,

$$S^{1,2} = S_2^1 = [A_1, A_1'] \quad \text{and} \quad S^{1,3} = S_1^1 = [A_1],$$

which therefore coincides with T_1^1; that is, $T_1^1 = S_1^1$, $T_2^1 = S_2^1$, $T_3^1 = S_3^1$ and $T_4^1 = S_4^1$. The tree determined by S (see Fig. 1) is now somewhat simpler than for class (a).

Fig. 2 will give us a schematic representation of a ruled surface S of class (b), if we let l_1', l_2', l_3', instead of l_1, l_2, l_3, be three neighboring generators of the surface; Q_1, Q_2, Q_3 will then coincide with P_1, P_2, P_3, respectively. The description given in § 46 for class (a) will apply without change, except that P_1, P_2 and P_3 can not now coincide with one another; that is, S_2^1 can not be a cone.

Hence, in order to construct the most general S_2^2 of class (b), we *first choose any curve* $S_1^1 (= T_1^1)$ *immersed in* F_4, which depends on three arbitrary functions. We *then choose* S_2^2 *to be any ruled surface whose generators pass through the points of* S_1^1 *and lie in its corresponding osculating planes* (generators of S_3^1), *with the single exception of the tangent surface* S_2^1 *of* S_1^1. This choice involves one additional arbitrary function, making four in all.

To illustrate class (b), let S_1^1 be the rational quartic

$$[A_1] = (1, \omega, \omega^2, \omega^3, \omega^4).$$

Then $S_3^1 = [A_1, A_1', A_1'']$, where

$$[A_1''] = (0, 0, 1, 3\omega, 6\omega^2).$$

Choosing $A_2 = A_1''$, we have $S_2^2 = [A_1, A_1'']$, which is a quintic surface. The orders of S_1^1, S_2^1, S_3^1 and S_4^1 are 4, 6, 6 and 4, respectively.

Class (c).

50. This case involves only three arbitrary functions. As in class (b), $S^{1,2}$ coincides with $T^{1,0}$, but is now a fixed line m (of range 0). That is,

$$S^{1,2} = S_2^0 = [A_1, A_1'] = T_2^0 = T^{1,0}.$$

S_2^2 may be described briefly as *a ruled surface immersed in* F_4, *whose generators all meet a fixed line m without passing through a fixed point.* In the extended tree (Fig. 1) T_1^1 no longer, as in class (b), belongs to the descending series determined by S_4^1; it is simply a point-range lying on m. S_3^1 is a line-cone or conical three-spread whose vertex is the line m.

In order to construct S_2^2 of class (c), we first choose any S_3^1 whose generating planes pass through a fixed line m; this involves one arbitrary function.

We then establish a one-to-one continuous correspondence between the planes of S_3^1 and the points of m; this involves one more arbitrary function. Finally, in each plane of S_3^1 we choose a line l passing through the corresponding point on m (and not coinciding with m), in such a way that the lines l form a continuous system; this involves still another arbitrary function, making three in all.

As an example of class (c), let

$$T_1^1 = [A_1] = (1, \omega, 0, 0, 0) \quad \text{and} \quad [A_2] = (0, 0, 1, \omega, \omega^2).$$

Then

$$S_2^2 = [A_1, A_2] = (1, \omega, \lambda, \lambda\omega, \lambda\omega^2),$$

which is a cubic surface; its generators connect the points of the conic $[A_2]$ with the corresponding points of the projective range $[A_1]$. Eliminating λ and ω, we find that S_2^2 is the partial intersection of the quadric three-spreads $x_1 x_5 = x_2 x_4$ and $x_1 x_4 = x_2 x_3$, which also intersect in the plane $x_1 = x_2 = 0$.

51. *The three fundamental classes of skew ruled surfaces S in F_4 may be distinguished by the fact that for class* (a) *every tangent to the curve T_1^1 at a point Q is distinct, in general, from the characteristic* (*focal line*) *of the tangent plane to S at Q, whereas for class* (b) *these two lines always coincide and are variable, and for class* (c) *they coincide and are fixed.*

We have shown that the tangent spread $S^{1,0}$ of an S_2^2 may be non-conical or may be a point-cone or line-cone, but *can not be a plane-cone*; that is, a pencil of three-flats. Moreover, T_1^1 may be a four-space curve, a three-space curve, or a range (straight line), but *can not be a plane curve.*

The projection of an S_2^2 from a point P of F_4 (see §§ 15–23) is in general a conical S_3^2 of type $(0\bar{2}0)$, illustrating the theorem of § 22. But if S_2^2 is of class (a_{21}) or (a_{22}) and P coincides with $S^{1,3} = S_1^0$, or if S_2^2 is of class (c) and P lies on the line $S^{1,2} = S_2^0$, then S_2^2 projects into $S^{1,1} = S_3^1$, illustrating the theorem of § 23.*

Non-Conical Skew Planar Three-Spreads in F_4.

52. The projective differential geometry of planar three-spreads could be obtained from that of ruled surfaces by the principle of duality. But it will be necessary for our purpose to build it on an independent foundation.

Let $S = S_3^2$ be a non-conical skew planar three-spread in F_4. Since its focal spread $S_{1,0}$ is of range 1, we see that a normal system of fundamental directrices ("P. D. G.," §§ 39–46) of S_3^2 is furnished by the two bottom rows of the array

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* For additional properties of S_2^2, see Ranum, *Trans. Amer. Math. Soc.*, Vol. XVI, p. 89, §§ 1–9.

$$\left.\begin{array}{cc} A_1''' & A_2''' \\ A_1'' & A_2'' \\ A_1' & A_2' \\ A_1 & \cdot \end{array}\right\} \tag{18}$$

That is,

$$S = S_3^2 = [A_1,\, A_1',\, A_2'],^*$$

from which we derive

$$S^{1,0} = S_5^0 = F_4 = [A_1,\, A_1',\, A_2',\, A_1'',\, A_2''],$$
$$S_{1,0} = S_1^1 = [A_1],$$
$$S_{1,1} = S_2^1 = [A_1,\, A_1'],$$

and

$$S_{1,2} = S_3 = [A_1,\, A_1',\, A_1''].$$

The most general transformation to new directrices, which preserves the normal character of the directrices, will be a special case of (b), § 2, "P. D. G."; namely:

$$\overline{A}_1 = \alpha A_1, \qquad \overline{A}_2' = \beta A_2' + \gamma A_1' + \delta A_1, \tag{19}$$

where $\alpha \neq 0$ and $\beta \neq 0$. By means of this transformation it is easy to see that the fundamental differential equations of the spread can be reduced to the canonical form

$$A_1''' + (l_1 A_2')' + l_2 A_2' + l_3 A_1 = 0, \tag{20}$$
$$A_2''' + \Sigma\,(k_{1i} A_i')' + \Sigma\, k_{2i} A_i' + k_3 A_1 = 0. \tag{20'}$$

We assume that this has been done.

53. Of all the ruled surfaces enclosed in S_3^2, the only developable is S_2^1; for A_1 is the only directrix of S_3^2 whose derivative is also a directrix of S_3^2. Hence, *every non-conical skew three-spread S in F_4 determines a unique developable surface whose generating lines lie in the generating planes of S, namely, its special enclosed spread $S_{1,1}$.*

54. The tangent three-flat to S_3^2 at a point

$$P = a A_1(\omega) + b A_1'(\omega) + c A_2'(\omega)$$

of the generator ω is the flat connecting the plane ω and the point

$$b A_1''(\omega) + c A_2''(\omega).$$

Being independent of the coefficient a, it is the same at all the points of the line joining P and the focal point $A_1(\omega)$ in the generator ω.

Defining a *focal line* of S_3^2 as any straight line lying in a generating plane and passing through the corresponding focal point, we see that *the tangent three-flats to S_3^2 at all the points of a focal line coincide*; hence, we may speak

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[*] The reason for writing A_2' with a prime is given in "P. D. G.," § 41.

of the tangent three-flats to S_3^2 *along its focal lines*;* there are ∞^2 focal lines and ∞^2 tangent three-flats. Obviously, the focal lines of an S_3^2 and the tangent planes to an S_2^2 are correlative in F_4.

Now consider those particular focal lines ($c = 0$) which are generators of S_2^1, or tangents to the focal curve S_1^1; the corresponding tangent three-flats to S_3^2 are generators of the spread

$$T = T_4^1 = [A_1, A_1', A_1'', A_2'],$$

which is the common enclosing spread of S and $S_{1,2}$; that is, they are the flats connecting the generating planes of S and the corresponding osculating planes of S_1^1. Hence, *among the ∞^2 tangent three-flats to a non-conical $S_3^2 = S$ there is a unique system T_4^1 of ∞^1 three-flats, such that the focal lines along which they are tangent to S_3^2 form a developable surface; T_4^1 is the common enclosing spread of S and $S_{1,2}$.*

By means of (20) we classify the spreads S_3^2 as follows:

Class (a): $l_1 \neq 0$; ∞^{5f_1} spreads.
Class (b): $l_1 = 0$, $l_2 \neq 0$; ∞^{4f_1} spreads.
Class (c): $l_1 = l_2 = 0$; ∞^{3f_1} spreads.

Class (a).

55. $S_{1,2}$ is of range 1 ($= S_3^1$), and

$$S_{1,3} = S_4 = [A_1, A_1', A_1'', (l_1 A_2')' + l_2 A_2'],$$

which is of range 1 or 0, depending on (20′). Moreover,

$$T_{1,0} = T_3^1 = [A_1, A_1', A_1'' + l_1 A_2'],$$

$$T_{2,0} = T_2^1 = \left[A_1, A_1'' + l_1 A_2' - \frac{l_2}{l_1} A_1'\right],$$

and

$$T_{3,0} = T_1 = \left[A_1'' + l_1 A_2' - \frac{l_2}{l_1} A_1' + \left(\left(\frac{l_2}{l_1}\right)' + \left(\frac{l_2}{l_1}\right)^2\right) A_1\right].$$

$T_{3,0}$ is of range 1 or 0, depending on (20). T_4^1, T_3^1 and T_2^1 enclose S_3^1, S_2^1 and S_1^1, respectively; but T_2^1 cannot coincide with S_2^1. Letting r and r' be the ranges of $S_{1,3}$ and $T_{3,0}$ respectively, we have the four subclasses:

(a_{11}), $r = 1$, $r' = 1$; (a_{12}), $r = 1$, $r' = 0$;
(a_{21}), $r = 0$, $r' = 1$; (a_{22}), $r = 0$, $r' = 0$.

Fig. 3 gives a schematic representation of S_3^2, when it belongs to the subclass (a_{11}) or (a_{21}). The planes π_1, π_2, π_3 are neighboring generators of S_3^2; P_1, P_2, P_3, P_4 are points of S_1^1; l_1, l_2, l_3 are generators of S_2^1; m_1, m_2, m_3, m_4

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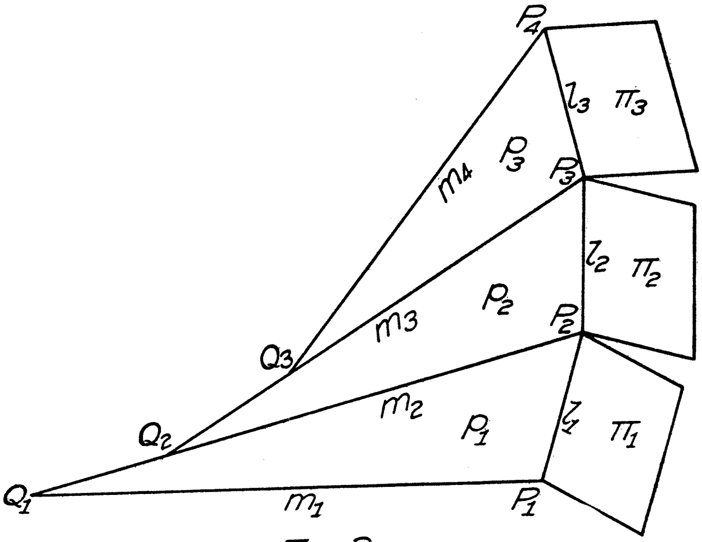
* The tangent *four-flats* to $S_3{}^2$ along its *generators* coincide with F_4 itself.

are generators of T_2^1; the planes $p_1 = (l_1, m_1)$, $p_2 = (l_2, m_2)$, $p_3 = (l_3, m_3)$ are generators of T_3^1; and Q_1, Q_2, Q_3 are points of T_1^1.

56. In constructing the general S_3^2 of class (a), we first choose any planar T_3^1 immersed in F_4, either non-conical or a point-cone, and then any developable surface S_2^1 enclosed in T_3^1, except its focal surface T_2^1. S_2^1 will be either immersed in F_4 or restricted to an F_3. Finally, we choose S_3^2 itself to be *any three-spread whose generating planes contain the generators of* S_2^1 *and lie in the tangent three-flats of* T_3^1, *with two exceptions, namely* T_3^1 *itself and the tangent spread* S_3^1 *of* S_2^1. These two are the only developable three-spreads enclosing S_2^1 and enclosed in T_4^1.



-FIG.3-

As an illustration of subclass (a_{11}), let

$$T_1^1 = [A] = (1, \omega, \omega^2, \omega^3, \omega^4),$$
$$S_1^1 = [A_1] = [A + \omega A'] = (1, 2\omega, 3\omega^2, 4\omega^3, 5\omega^4),$$

and

$$S_3^2 = [A_1, A_1', A_2'],$$

where $[A_1'] = (0, 1, 3\omega, 6\omega^2, 10\omega^3)$ and $[A_2'] = [A'''] = (0, 0, 0, 1, 4\omega)$. S_3^2 is easily seen to be of order 5.

Class (b).

57. $S_{1,2}$ is again of range 1, and

$$S_{1,3} = S_4^1 = [A_1, A_1', A_1'', A_2'],$$

which therefore coincides with T_4^1. Fig. 3 can be made to represent this case by letting the generating planes π_1, π_2, π_3 of S_3^2 pass through the lines m_2, m_3, m_4 instead of l_1, l_2, l_3; they lie in the tangent three-flats of T_3^1, as before.

In constructing the general S_3^2 of class (b), we choose a proper developable T_3^1, and for generators of S_3^2 take *any continuous system of planes containing the focal lines of* T_3^1 *and lying in the corresponding tangent three-flats, with the exception of* T_3^1 *itself.*

Class (c).

58. $S_{1,2}$ is now of range 0 (a fixed plane π), and coincides with $T_{1,0}$; that is,

$$S_{1,2} = S_3^0 = [A_1, A_1', A_1''] = T_3^0 = T_{1,0}.$$

S_1^1 is a plane curve lying in π, and T_4^1 is a pencil of three-flats whose vertex is π. S_3^2 may be described as *a non-conical three-spread immersed in* F_4, *whose generating planes intersect a fixed plane* π *in straight lines* (instead of points). These straight lines are the tangents to the focal curve S_1^1.

59. Classes (a), (b) and (c) may be characterized briefly as follows: Class (a): $S_{1,2} = S_3^1$ does not coincide with $T_{1,0} = T_3^1$. Class (b): S_3^1 coincides with T_3^1. Class (c): $S_{1,2} = S_3^0$ coincides with $T_{1,0} = T_3^0$.

We have shown that the focal curve S_1^1 of an S_3^2 may be a plane curve, a three-space curve, or a four-space curve, but *can not be a straight line* (point-range); and that T_4^1 may be non-conical, or may be a point-cone or plane-cone (pencil of three-flats), but *can not be a line-cone.*

A three-flat Q of F_4 (see §§ 15–23) will intersect an S_3^2 in a ruled surface U, which is in general an S_2^2 (§ 22). But if S_3^2 is of class (a_{21}) or (a_{22}) and Q coincides with $S_{1,3} = S_4^0$, or if S_3^2 is of class (c) and Q contains the plane $S_{1,2} = S_3^0$, then U coincides with $S_{1,1} = S_2^1$ (§ 23).*

Irregular Ruled Surfaces Immersed in F_5.

60. Coming now to five-dimensional space, we have three irregular classes of spreads to consider, namely ($0\bar{2}110$), ($011\bar{2}0$) and ($01\bar{2}10$), of which the first comprises the skew ruled surfaces whose tangent four-spreads are developable.

Let $S = S_2^2 = [A_1, A_2]$ be any such spread. Then

$$S^{1,0} = S_4^1 = [A_1, A_2, A_1', A_2'].$$

The fundamental differential equations can be chosen in the canonical form

$$A_1'' + (l_1 A_2)' + l_2 A_2 = 0,$$
$$A_2^{(4)} + (k_1 A_2)''' + (k_2 A_2)'' + \Sigma (k_{3i} A_i)' + \Sigma k_{4i} A_i = 0.$$

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* For additional properties of S_3^2, see Ranum, *loc. cit.*, §§ 10–14.

Hence,

$$S^{2,0} = S_5^1 = [A_1, A_2, A_1', A_2', A_2'']$$

and

$$S^{3,0} = S_6^0 = F_5 = [A_1, A_2, A_1', A_2', A_2'', A_2'''].$$

Exactly as in the case of ruled surfaces in F_4 (§ 42) we have $S^{1,1} = S_3^1 = [A_1, A_2, A_1']$ and $S^{1,2} = S_2 = [A_1, A_1' + l_1 A_2]$; and we define $T_1^1 = [A_1]$ as the curve of intersection of S and $S^{1,2}$.

The classification of these spreads in F_5 is precisely the same as in F_4, the number of arbitrary functions for each class being one greater than for the corresponding class in F_4. In the description of each class the only changes are the following: In class (a) S_3^1 is immersed in F_5, and T_1^1 is immersed either in F_5 or in an F_4; in class (b) T_1^1 ($= S_1^1$) is immersed in F_5; in class (c) S_3^1 is a line-cone, as before, but is now immersed in F_5 instead of F_4.

Non-conical, Irregular Four-spreads in F_5.

61. In F_5 the four-spreads generated by ∞^1 three-flats are dual to the ruled surfaces. We now consider the class $(011\bar{2}0)$ of four-spreads S whose focal surfaces are developable (and non-conical). They are ∞^{6f_1} in number. Expressing S in terms of normal directrices, we have

$$S = S_4^2 = [A_1, A_1', A_1'', A_2''].$$

Then

$$S_{1,0} = S_2^1 = [A_1, A_1'],$$

$$S_{1,1} = S_3^1 = [A_1, A_1', A_1''],$$

and

$$S_{1,2} = S_4 = [A_1, A_1', A_1'', A_1'''].$$

We define T as the common enclosing spread of S and $S_{1,2}$; that is,

$$T = T_5^1 = [A_1, A_1', A_1'', A_1''', A_2''].$$

The three classes of spreads may be described briefly as follows:

Class (a): T_5^1 is non-conical or a point-cone, and $S_{1,2} = S_4^1$ is immersed in F_5 or in an F_4 and is enclosed in T_5^1, but does not coincide with $T_{1,0} = T_4^1$. The generating three-flats of S_4^2 pass through the generating planes of S_3^1 and lie in the generating four-flats of T_5^1.

Class (b): T_5^1 is non-conical. The generating three-flats of S_4^2 lie in generating four-flats of T_5^1 and pass through the generating planes of $T_{2,0} = T_3^1 = S_3^1$.

Class (c): T_5^1 is a pencil of four-flats, whose vertex is a three-flat F. $S_{2,0} = S_1^1$ is a curve immersed in F. The generating three-flats of S_4^2 pass through the osculating planes of S_1^1 and lie in the corresponding four-flats of T_5^1. S_4^2 may be described as a non-conical four-spread whose generating three-flats meet a fixed three-flat in planes.

Proper Planar Three-Spreads of Range 2 in F_5.

62. We now come to the last and most interesting class of spreads in F_5, namely the self-dual class $(01\bar{2}10)$, which comprises the non-conical, irregular planar three-spreads immersed in F_5. This is the simplest class of m-spreads in F_n having two branch series, one ascending and one descending. The array (18), § 52, is again fundamental, and we have

$$S = S_3^2 = [A_1, A_1', A_2'].$$

Then

$$\begin{aligned} S_{1,0} &= T_1^1 = [A_1], \\ S_{1,1} &= T_2^1 = [A_1, A_1'], \\ S_{1,2} &= T_3 = [A_1, A_1', A_1''], \end{aligned}$$

and

$$S^{1,0} = R_5^1 = [A_1, A_1', A_2', A_1'', A_2''].$$

The fundamental differential equations are

$$\left.\begin{aligned} G &\equiv \Sigma\, g_{0i} A_i''' + \Sigma\, g_{1i} A_i'' + \Sigma\, g_{2i} A_i' + g_{31} A_1 = 0, \\ H &\equiv \Sigma\, h_{0i} A_i^{(4)} + \Sigma\, h_{1i} A_i''' + \Sigma\, h_{2i} A_i'' + \Sigma\, h_{3i} A_i' + h_{41} A_1 = 0, \end{aligned}\right\} \qquad (21)$$

where

$$\begin{vmatrix} g_{01} & g_{02} \\ h_{01} & h_{02} \end{vmatrix} \neq 0.$$

The most general transformation to new normal directrices is again (19), and the most general transformation of the equations (21) themselves is

$$\bar{G} = \lambda G, \qquad \bar{H} = \mu G' + \nu H,$$

where $\lambda \neq 0$ and $\nu \neq 0$. When (21) is simplified by means of these transformations, the result depends essentially on whether g_{02} is equal to zero or not. If $g_{02} \neq 0$, we obtain class (α); and if $g_{02} = 0$, classes (β), (γ) and (δ).

Class (α).

63. This is the general class, containing ∞^{7f_1} spreads. The differential equations (21) are reducible to the canonical form

$$A_2''' + (l_1 A_1)' + l_2 A_2' + l_3 A_1 = 0 \qquad (22)$$

and

$$A_1^{(4)} + \Sigma\, (k_{2i} A_i')' + \Sigma\, k_{3i} A_i' + k_{41} A_1 = 0. \qquad (22')$$

It follows that

$$\begin{aligned} S_{1,2} &= T_3^1, \\ S_{1,3} &= T_4 = [A_1, A_1', A_1'', A_1'''], \\ S^{1,1} &= R_4^1 = [A_1, A_1', A_2', A_2''], \\ S^{1,2} &= R_3^1 = [A_1, A_2', A_2''], \end{aligned}$$

and

$$S^{1,3} = R_2 = [A_2', A_2'' + l_1 A_1].$$

Geometrically, class (α) is distinguished from the other classes by the fact that $S_{1,1}\,(=T^1_2)$, which is in every case enclosed in $S^{1,1}=R^1_4$, is not in this case enclosed in $S^{1,2}=R^1_3$ (see Fig. 4). Hence, in order to construct the most general S^2_3 of class (α), we can first choose any developable three-spread R^1_3 immersed in F_5, either non-conical, a point-cone or a line-cone; we can then choose any curve T^1_1 situated on R^1_3, but not on its focal surface (or vertex) R_2; T^1_1 will be a five-space, four-space or three-space curve. Finally, we can choose S^2_3 itself to be *any three-spread (without exception) whose generating planes pass through the tangents to T^1_1 and lie in the corresponding tangent three-flats to R^1_3.*

64. Coming to the extended tree, we define R as the common enclosed spread, or surface of intersection of S^2_3 and R^1_3. Then $R=R^2_2=[A_1, A'_2]$, $R^{1,0}=R^1_4=S^{1,1}$, $R^{1,1}=R^1_3=S^{1,2}$, etc. R is a non-regular ruled surface. As we have seen (§ 60), its structure depends essentially on that of the curve $U=U^1_1=[A'_2]$, in which it is intersected by $R^{1,2}=R_2$. *R can belong to any one of the three classes (six subclasses) of such surfaces.* Thus it belongs to class (a), if in equation (22) $l_1\neq 0$; to class (b), if $l_1=0$ and $l_3\neq 0$; to class (c), if $l_1=l_3=0$.

Dually, we define T as the common enclosing spread of S^2_3 and T_3; that is,

$$T=T^2_4=[A_1, A'_1, A''_1, A'_2],\quad T_{1,0}=T^1_2=S_{1,1},\quad T_{1,1}=T^1_3=S_{1,2},\ \text{etc.}$$

T is a non-conical, irregular four-spread, and its structure (see § 61) depends on that of the five-spread

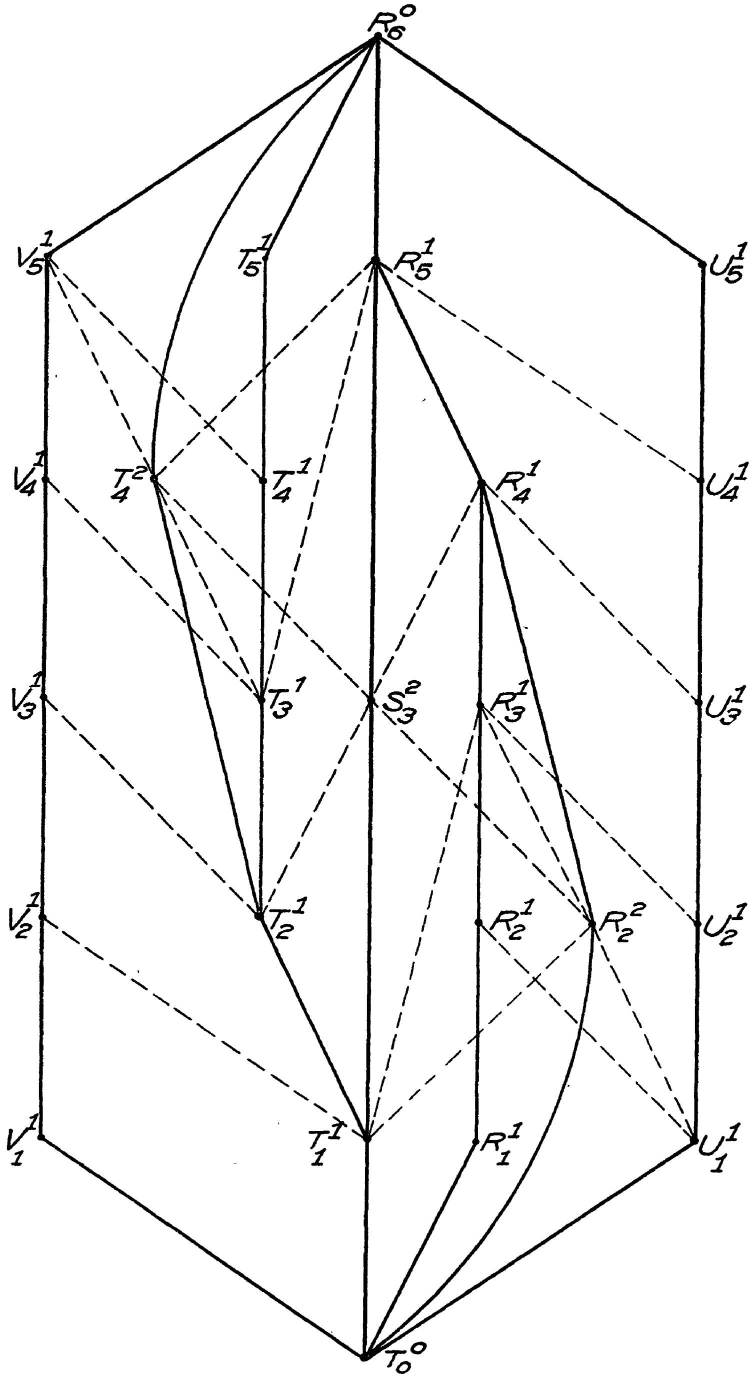
$$V=V^1_5=[A_1, A'_1, A''_1, A'''_1, A'_2],$$

which is the common enclosing spread of T and $T_{1,2}=T_4$. *T can belong to any one of the three classes (six subclasses) of such spreads.* If in equation (22′) $k_{22}\neq 0$, T belongs to class (a); if $k_{22}=0$ and $k_{32}\neq 0$, to class (b); and if $k_{22}=k_{32}=0$, to class (c).

The class to which T^2_4 belongs is obviously independent of that to which R^2_2 belongs; this independence can easily be shown to extend to the subclasses (a_{11}), etc., as well. The spreads S^2_3 of class (α) can therefore be divided into $6\cdot 6=36$ subclasses, depending on the subclasses to which R^2_2 and T^2_4 belong. If R^2_2 belongs to subclass (a_{11}) and T^2_4 to subclass (b), we shall say that S^2_3 belongs to the subclass ($\alpha_{a_{11}b}$); and similarly for the other combinations.

It is not difficult to verify that the entire extended tree determined by S^2_3 is made up of the spreads already mentioned, together with their successive tangent and focal spreads. In the case of the most general subclass of all, namely ($\alpha_{a_{11}a_{11}}$), the extended tree is indicated in Fig. 4, where the full and dotted lines have the same significance as in Fig. 1.

65. By means of the extended tree we can construct the most general spread S_3^2 of class (α), for which R_2^2 belongs to any prescribed subclass, as follows. We first choose a developable three-spread R_3^1 immersed in F_5, either non-conical, a point-cone or a line-cone, according as R_2^2 is to belong to one of the subclasses (a_{11}), (a_{12}), (b), to one of the subclasses (a_{21}), (a_{22}), or to (c); this choice depends on four, three or two arbitrary functions, respectively.



—FIG. 4—

We next choose on its focal surface (or fixed line) R_2 a five-space or four-space curve (or point-range) U_1^1, which depends on either one additional arbitrary function or none, according as R_2^2 belongs to one of the subclasses (a_{11}), (a_{21}), (c) or not.

The third and final step is to choose *a curve* T_1^1, *situated on* R_3^1, *but not on its focal spread* R_2 *or on the tangent spread* U_2^1 *of* U_1^1. Any further specialization in the choice of T_1^1 has no effect on the subclass to which R_2^2 belongs, but does affect the subclass to which T_4^2 belongs. Thus if T_1^1 is a five-space curve and depends, therefore, on two more arbitrary functions, T_4^2 will belong to one of the subclasses (a_{11}), (a_{12}), or (b); if T_1^1 is a four-space curve and involves one more arbitrary function, T_4^2 will belong to (a_{21}) or (a_{22}); finally, if T_1^1 is a three-space curve, involving no more arbitrary functions, T_4^2 will belong to (c).

The curves U_1^1 *and* T_1^1, *as so chosen, determine completely the original spread* S_3^2 and the entire extended tree. For R_2^2 is generated by the lines joining corresponding points of U_1^1 and T_1^1, and S_3^2 is generated by the planes joining the points of U_1^1 and the corresponding tangents to T_1^1. Moreover, T_4^2 is generated by the three-flats joining the points of U_1^1 and the corresponding osculating planes of T_1^1, and V_5^1 is generated by the four-flats joining the points of U_1^1 and the corresponding osculating three-flats of T_1^1.

66. Dually, it is clear that by properly choosing T_3^1, V_5^1 and R_5^1, we can construct the most general S_3^2 for which T_4^2 belongs to any prescribed subclass. The four-flats of V_5^1 intersect the corresponding four-flats of R_5^1 in the three-flats of T_4^2; and they intersect the three-flats of R_4^1 in the planes of S_3^2, the planes of R_3^1 in the lines of R_2^2, and the lines of R_2 in the points of U_1^1.

67. The degree of generality of each of the 36 subclasses of (α) is now easy to compute; the number of arbitrary functions varies from seven to three. At one extreme is $(\alpha_{a_{11}a_{11}})$, which contains ∞^{7f_1} spreads; while at the other extreme are $(\alpha_{a_{22}a_{22}})$, $(\alpha_{a_{22}c})$, $(\alpha_{ca_{22}})$ and (α_{cc}), each of which contains ∞^{3f_1} spreads.

S_3^2 is obviously determined not merely by U_1^1 and T_1^1, or by V_5^1 and R_5^1, but also by R_2^2 and T_2^1, by T_4^2 and R_4^1, or self-dually by R_2^2 and T_4^2. This is the first case we have met of *an irregular spread that is completely determined by the other spreads of its extended tree.*

It is to be noticed, however, that if R_2^2 is chosen arbitrarily, it is in general impossible to find an S_3^2, for which T_4^2 shall belong to one of the subclasses (a_{22}) or (c), having the lowest degree of generality. This is easily seen, for instance, in the case where R_2^2 belongs to (a_{11}) and T_4^2 to (c). For then R_2^2 involves six arbitrary functions, whereas S_3^2 has only five functions at its disposal. Geometrically, also, we see that when T^2_4 belongs to (c), R^2_2 can not be the most general spread of class (a_{11}); for T^1_1 is then immersed in an F_3 and is therefore the curve of intersection of F_3 and R^2_2. But if R^2_2 is chosen arbitrarily, its generators will not meet any F_3.

Classes (β), (γ) *and* (δ).

68. Returning to the differential equations (21), we now take up the case in which $g_{02} = 0$. They are no longer reducible to the form (22), (22′), but instead to the form

$$A_1''' + (l_1 A_2')' + l_2 A_2' + l_3 A_1 = 0, \tag{23}$$

$$A_2^{(4)} + \Sigma\, (k_{2i} A_i')' + \Sigma\, k_{3i} A_i' + k_{41} A_1 = 0. \tag{23'}$$

It follows that

$$S^{1,1} = R^1_4 = [A_1,\ A_1',\ A_1'',\ A_2'],$$

and

$$S^{1,2} = R_3 = [A_1,\ A_1',\ A_1'' + l_1 A_2'].$$

The further classification depends on the values of l_1 and l_2 in (23), as follows: Class (β), $l_1 \neq 0$; class (γ), $l_1 = 0$, $l_2 \neq 0$; class (δ), $l_1 = l_2 = 0$. It is evident that these three classes correspond exactly to classes (a), (b) and (c) of the skew planar three-spreads in F_4, as given in §§ 55–59. The only essential difference is that the rôle of the spread T^1_4 is here played by $S^{1,1} = R^1_4$. The number of arbitrary functions for each class is one greater than for the corresponding class in F_4, and is therefore the same as for the corresponding class of ruled surface of type ($0\bar{2}110$) in F_5, or of four-spreads of type ($011\bar{2}0$), as given in §§ 60, 61.

69. Classes (β), (γ) and (δ) may be described briefly as follows:

Class (β), four subclasses: $S^{1,2} = R^1_3$ is non-conical or a point-cone. $S_{1,2} = T^1_3$ is immersed in F_5 or in an F_4. R^1_3 encloses T^1_2, but does not coincide with T^1_3. The number of arbitrary functions varies from six to four. The subclass (β_{11}) or (β_{21}) is represented in Fig. 3.

Class (γ): R^1_3 is a non-conical spread immersed in F_5 and coincides with T^1_3. There are five arbitrary functions involved.

Class (δ): $S^{1,2} = R^0_3$ coincides with $S_{1,2} = T^0_3$. T^1_1 is a plane curve; R^1_4 is a plane-cone. S^2_3 may be described as *a proper three-spread whose generating planes meet a fixed plane in lines.* There are four arbitrary functions involved.

The spreads R and T, as defined in § 64 for the class (α), become coincident, for the classes (β), (γ) and (δ), with T^1_2 and R^1_4, respectively. Hence, the extended tree, Fig. 4, reduces to the primary tree determined by S^2_3.

70. The following general theorems concerning three-spreads S^2_3 of type ($01\bar{2}10$) follow almost immediately from what has been proved.

Among the ruled surfaces, ∞^{2f_1} in number, which are enclosed in S_3^2, those whose generators are not focal lines of S_3^2 are all skew surfaces.*

Among those whose generators *are* focal lines of S_3^2, ∞^{f_1} in number, there is *one exceptional surface that is developable*, viz., its special enclosed spread T_2^1.

Among the latter there is *also an exceptional surface R having the property that the tangent three-flats to S_3^2 along the generators of R form a developable four-spread R_4^1.*

Among the four-spreads, ∞^{2f_1} in number, which enclose S_3^2, those whose generating three-flats are not tangent to S_3^2 are all skew four-spreads.

Among those whose generating three-flats *are* tangent to S_3^2, ∞^{f_1} in number, there is *one exceptional four-spread that is developable*, viz., its special enclosing spread R_4^1.

Among the latter there is *also an exceptional four-spread T having the property that the focal lines of S_3^2 along which the generators of T are tangent to S_3^2 form a developable surface T_2^1.*

If S_3^2 belongs to class (β), (γ) or (δ), R coincides with T_2^1 and T with R_4^1.

71. *Among the ruled surfaces generated by focal lines of S_3^2, T_2^1 and R are the only ones which are not of type* $(0\bar{2}20)$. If S_3^2 is of class (α), R is of type $(0\bar{2}110)$; its tangent spread R_4^1 is developable.

Among the four-spreads whose generators are tangent to S_3^2, R_4^1 and T are the only ones which are not of type $(02\bar{2}0)$. If S_3^2 is of class (α), T is of type $(011\bar{2}0)$; its focal surface T_2^1 is developable.

72. *If S_2^2 is a skew ruled surface generated by focal lines of S_3^2, the tangent three-flats to S_3^2 along the generators of S_2^2 are also the tangent three-flats to S_2^2 itself,* and therefore generate its tangent spread.

If S_4^2 is a skew four-spread whose generators are tangent to S_3^2, the focal lines of S_3^2 along which the generators of S_4^2 are tangent to S_3^2 are also the focal lines of S_4^2 itself, and therefore generate its focal surface.

If S_3^2 is of class (α), there exist ∞^{f_1} three-spreads enclosed in T_4^2 and enclosing R_2^2, *all of which are of range* 3 *except S_3^2 itself.*

73. The spreads S_3^2 of class (α), for which R_2^2 is of subclass (c), are precisely those proper three-spreads of range 2 whose generating planes meet a fixed line R_2^0, but do not meet a fixed plane in lines.

The spreads S_3^2 of class (α), for which T_4^2 is of subclass (c), are precisely those proper three-spreads of range 2 whose generating planes meet a fixed three-flat T_4^0 in lines, but do not meet a fixed plane in lines.

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* See § 54.

The subclass (α_{cc}) consists of the proper three-spreads of range 2 whose *generating planes meet a fixed line* R_2^0 *and also meet a fixed three-flat* T_4^0 *in lines.* R_2^0 and T_4^0 will necessarily be non-intersecting.

To illustrate this case, let

$$T_1^1 = [A_1] = (0, 0, 1, \omega, \omega^2, \omega^3)$$

and

$$U_1^1 = [A_2'] = (1, \omega, 0, 0, 0, 0).$$

Then, R_2^0 is the line $x_3 = x_4 = x_5 = x_6 = 0$, and T_4^0 is the three-flat $x_1 = x_2 = 0$. T_1^1 is a curve immersed in T_4^0. R_3^1 is a line-cone generated by the planes connecting R_2^0 with the points of T_1^1. T_1^1 and R_3^1 are of order 3, T_2^1 and R_4^1 are of order 4, T_3^1 and R_5^1 are of order 3. The lines of R_2^2 connect the points of the range U_1^1 with the corresponding points of T_1^1, the planes of S_3^2 connect the points of U_1^1 with the corresponding tangents to T_1^1, and the three-flats of T_4^2 connect the points of U_1^1 with the corresponding osculating planes of T_1^1. The orders of R_2^2, S_3^2 and T_4^2 are 4, 5 and 4, respectively.

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